Forecasting of turbulence and mountain waves for aviation meteorology purposes

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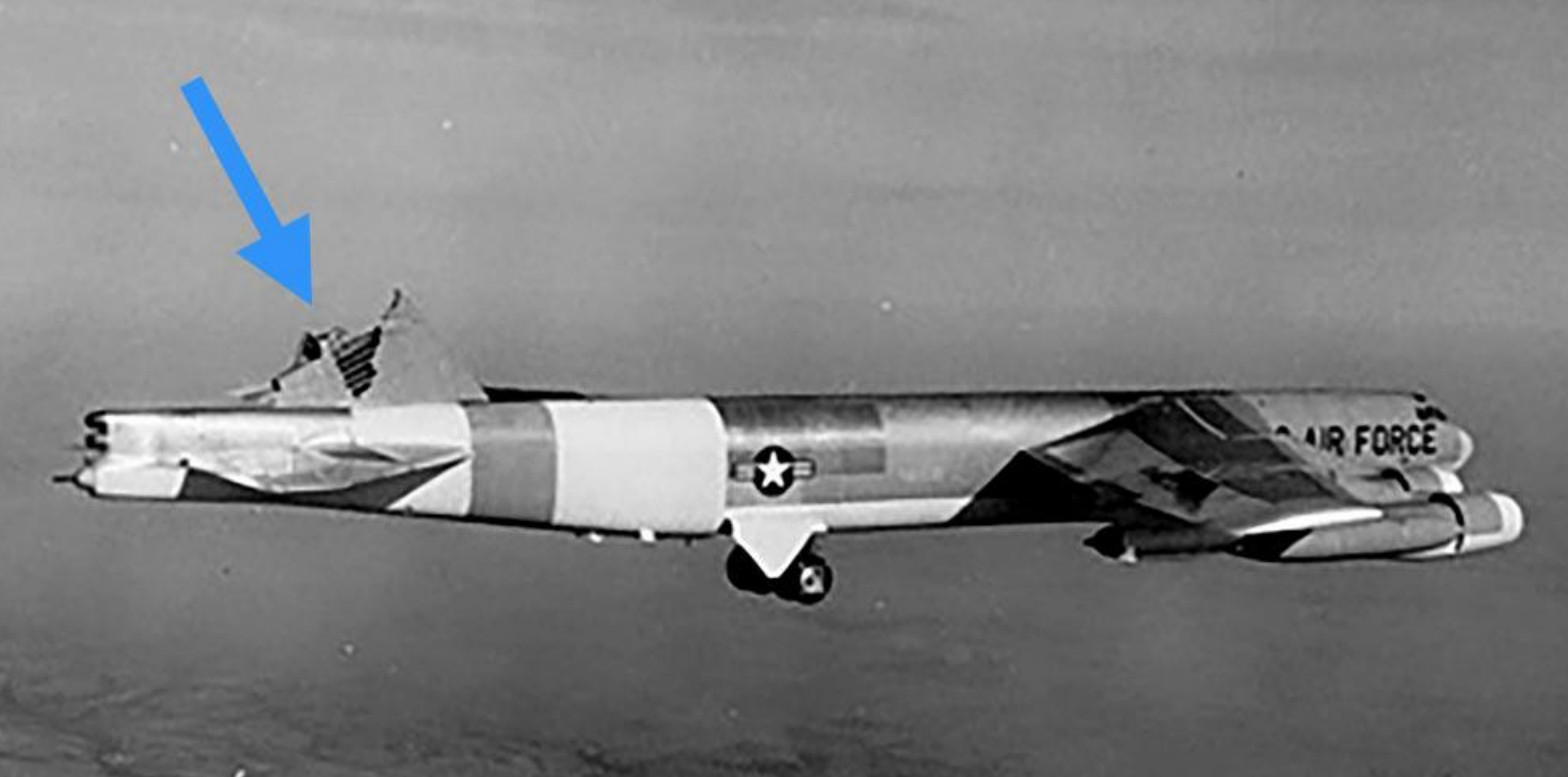
Photo: Kornél Kolláth

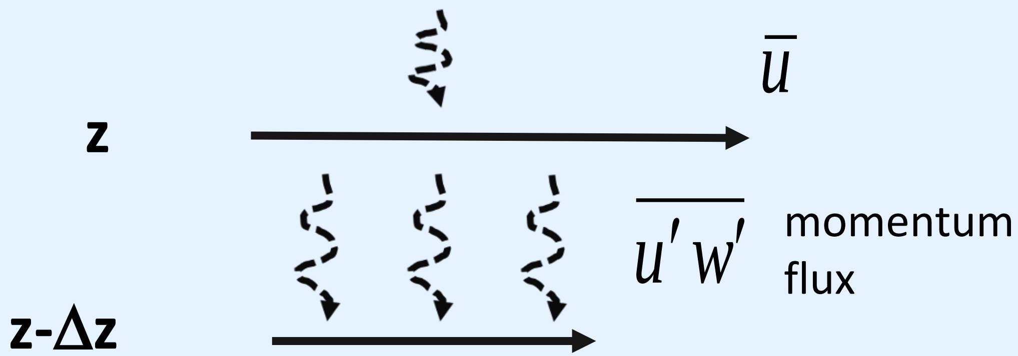
Effects of mountain wave flow and turbulence



B-52H after catching SEV MTW (landed ‘safely’).
East Spanish Peak, Colorado, USA, 1964.01.10.
http://www.whiteeagleaerospace.com/an-amazing-tail/

Aircraft turbulence - definition

- „Irregular motion of an aircraft in flight, especially when characterized by rapid up-and-down motion, caused by a rapid variation of atmospheric wind velocities" (AMS Glossary)
- Microscale atmospheric turbulence in general: characterized by fluctuations of all three velocity components and also other parameters (e.g. air pressure), provides both horizontal and vertical transport of the fluid properties. These cause variation of the property in time.



$$u = \bar{u} + u' \qquad v = \bar{v} + v' \qquad w = \bar{w} + w' \qquad \frac{D\bar{u}}{Dt} \approx -\frac{\partial \overline{u'w'}}{\partial z}$$

- Fluctuations are related to vertical + horizontal wind shear and static stability (buoyancy)
- In operational NWP models the fluxes are mostly parameterized (typical scales of the eddies are < 1km)

Turbulent kinetic energy (TKE)

- Widely used in NWP model parameterizations

$$e = \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right) \quad units : \left[m^2\, s^{-2}\right]$$

- It can be calculated from a prognostic equation

$$\frac{d\,e}{d\,t} = -\overline{u'\,w'}\,\frac{\partial \bar{u}}{\partial z} - \overline{v'\,w'}\,\frac{\partial \bar{v}}{\partial z} + \frac{g}{\overline{\theta}}\left(w'\,\theta'\right) - \frac{\partial\,\overline{w'\,e}}{\partial z} - \frac{1}{\rho}\,\frac{\partial\,\overline{w'\,p'}}{\partial z} - \varepsilon$$

Wind shear term

Buoyancy term

Turbulent transport of TKE

Pressure-correlation term

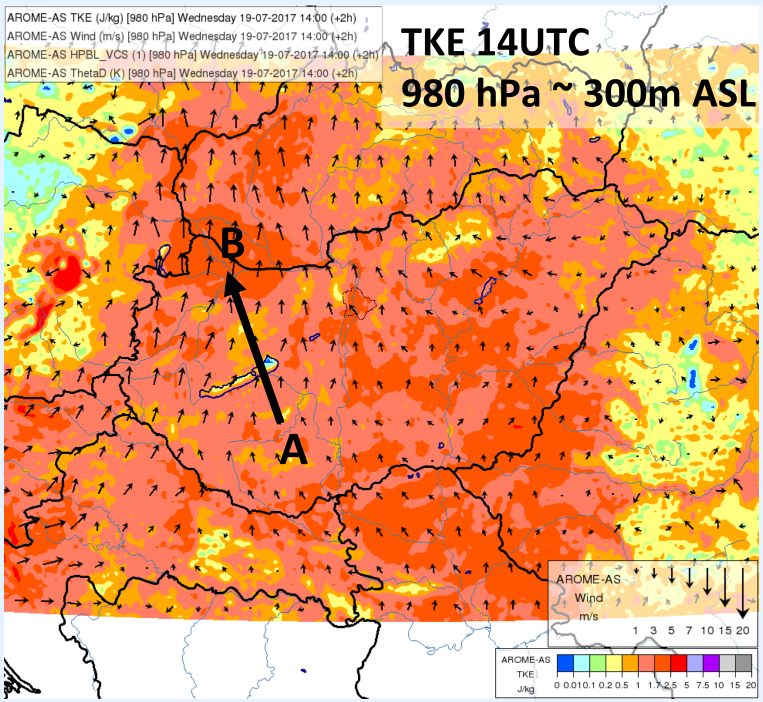
TKE dissipation

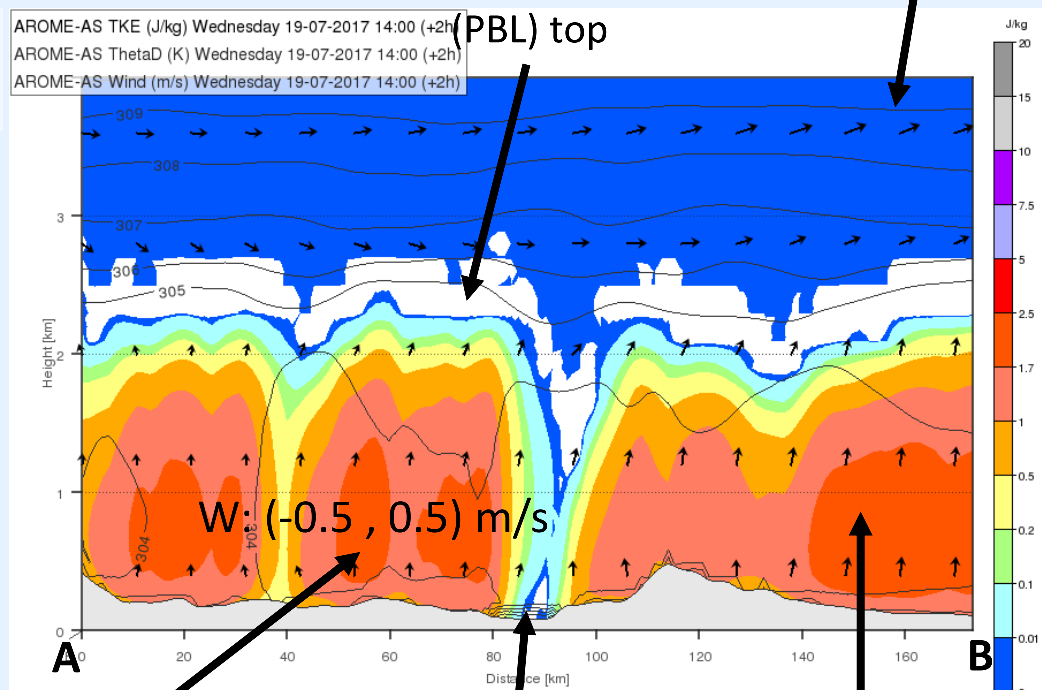
TKE forecasts in usual conditions (example)

- TKE from the AROME model (2.5 km horizontal resolution)
- Mixed (convective) boundary layer
- Strong convective motions („thermals”, „plumes”) can appear

Stable flow over the PBL

Planetary Boundary Layer (PBL) top





Moderate TKE
1-2 J/kg

Statically stable region over the Lake Balaton

Mixed region, almost dryadiabatic lapse-rate

TKE expressed in different units

- In NWP models, 10m wind gusts are also parameterized upon TKE
- Similar relationship can be used for the levels above the surface as well
- TKE is thus represented as a fluctuation of wind speed (either vertical or horizontal) with corresponding kinetic energy

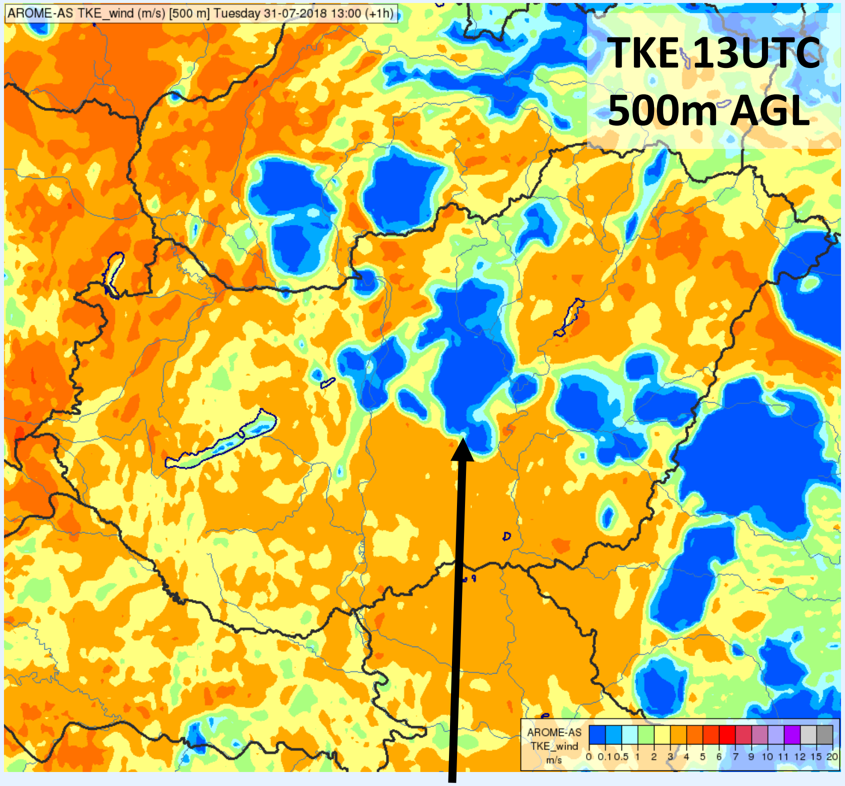
$$TKE = 0.5\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \approx \frac{3}{2} u'^2$$

$$u' \approx \alpha\left(\sqrt{2/3}\right)\sqrt{TKE}\ , \alpha = 3.5$$

α is probably a function of height

Usual conditions: u’ is in order of several m/s (**0-5 m/s**)

Climb rate of small-weighted aircrafts is of similar magnitude

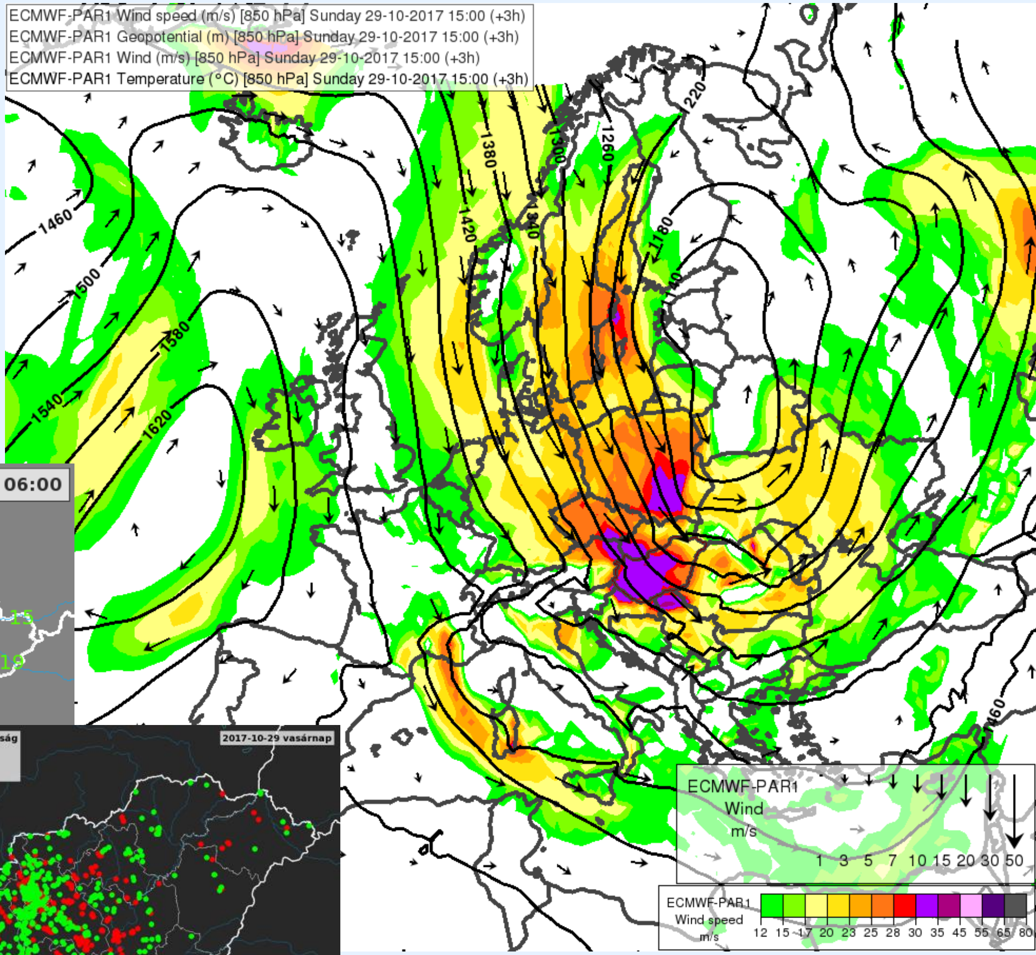


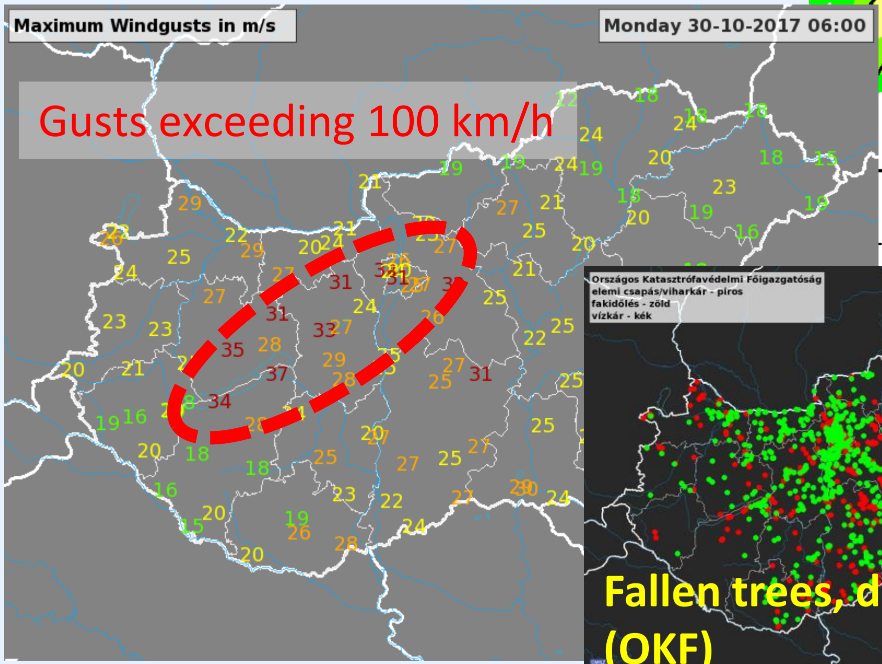
Thunderstorm cold pools: stable flow, low TKE

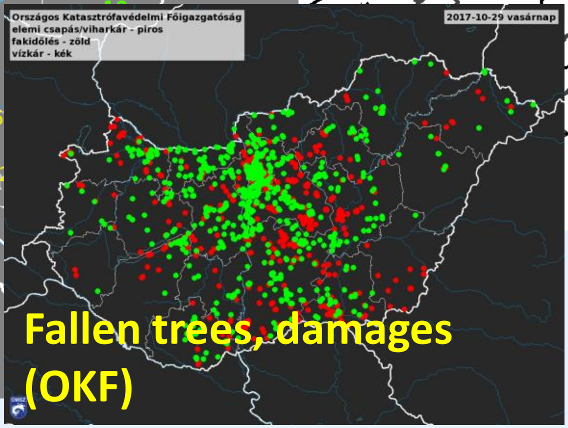
Strong turbulence: 29 October 2017 windstorm

- Lowering of the jet on the rear side of a deep cyclone
- Deep convection formed close to a cold front
- Wind gusts up to 130 km/h registered, extensive damage

AT 850 hPa+wind speed

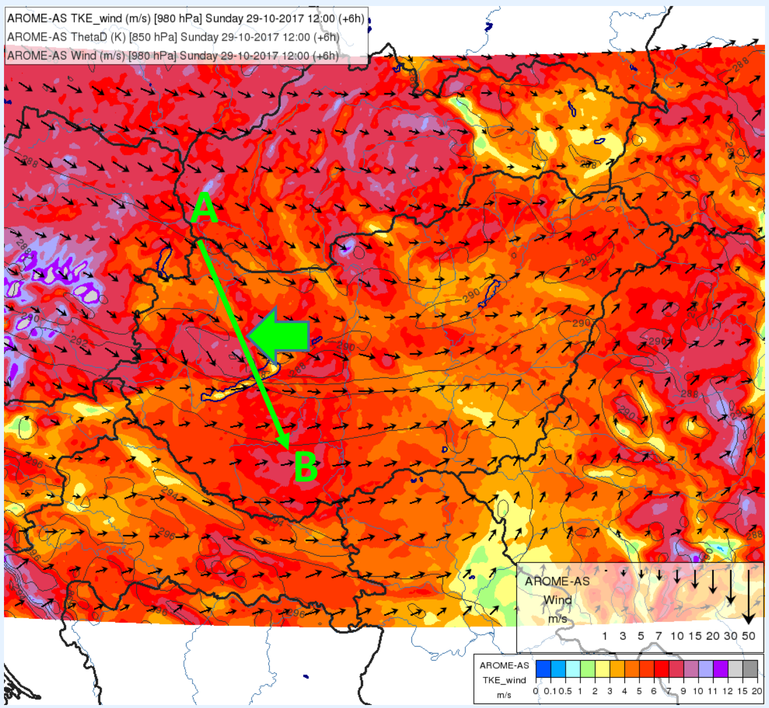




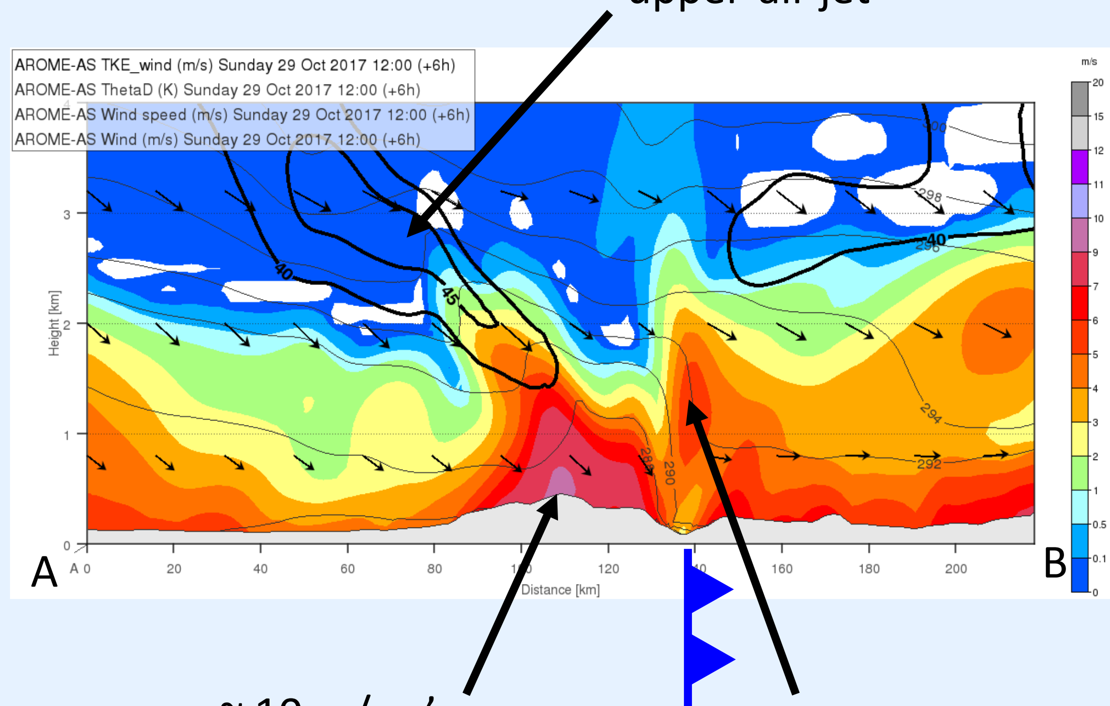


TKE profile

- Magnitude of u’ was about 10 m/s (TKE was 10-11 J/kg)
- Area of high turbulence reached the lowered jet area
- Weakly stable region behind the cold front, not related to deep convection



Lowering of the upper-air jet



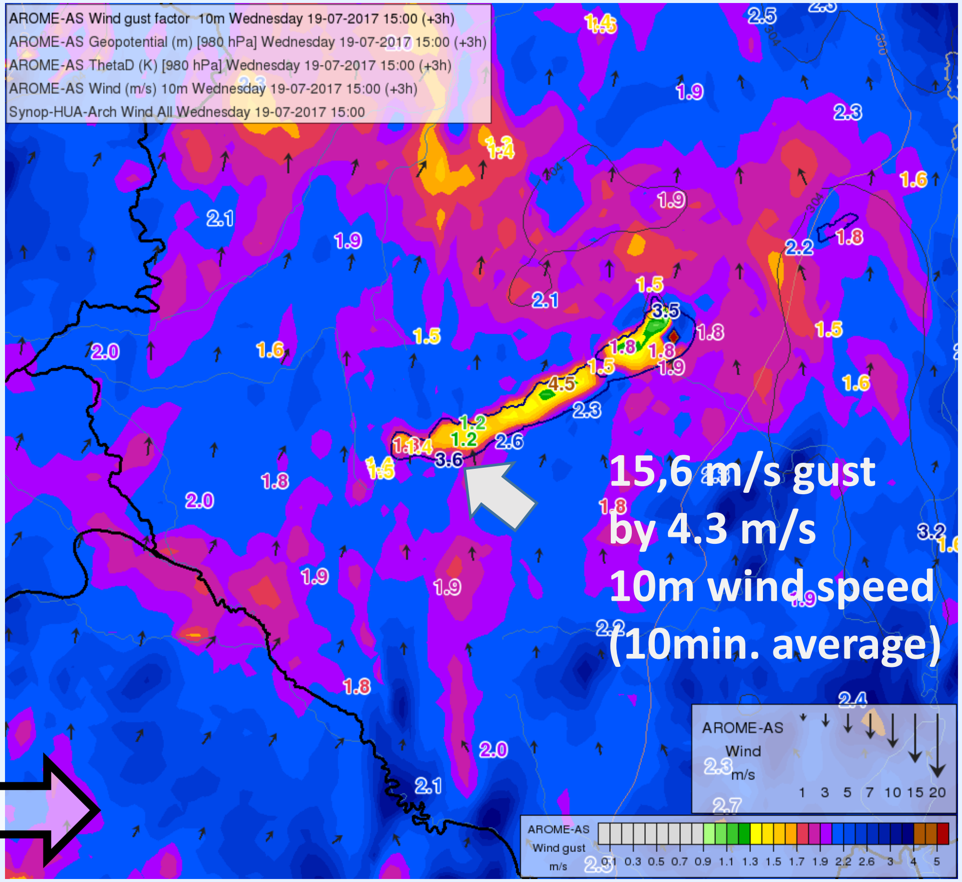
~ 10 m/s u’
> 10 J/kg TKE

Deep convection
w > 3 m/s

Turbulence detection, estimation

- Wind gusts and gust factors at 10m : footprints of turbulence at the ground
- Mostly 1.2-1.8. In neutral/unstable conditions often 1.8-3
- Aircraft observations (e.g. accelerometers), reports (PIREP, ARS, AMDAR) – medium/heavy size aircrafts
- Remote measurements (windprofiler, lidar) – rare in central Europe

Gust factor: wind gust/10m wind speed



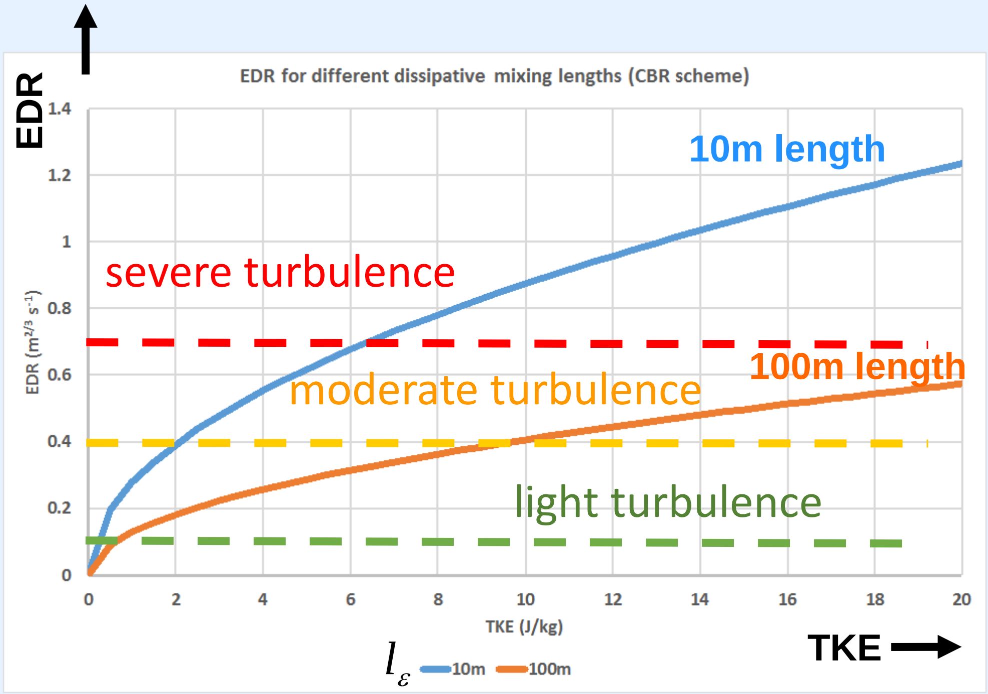
Extraordinary high gust factors (> 3) caused by local, strong turbulence are usually not predicted by operational NWP models

EDR diagnostics

- ICAO, Annex 3 turbulence metric – categories of turbulence defined upon the cube root of the Eddy Dissipation Rate (EDR)

$$EDR = \varepsilon^{1/3}$$

- Related to TKE and size of the turbulent eddies (mixing length l_ε)
- Aircraft independent parameter, but thresholds are different for heavy- and light-weight aircrafts



(ICAO thresholds refer to medium-weight aircrafts)

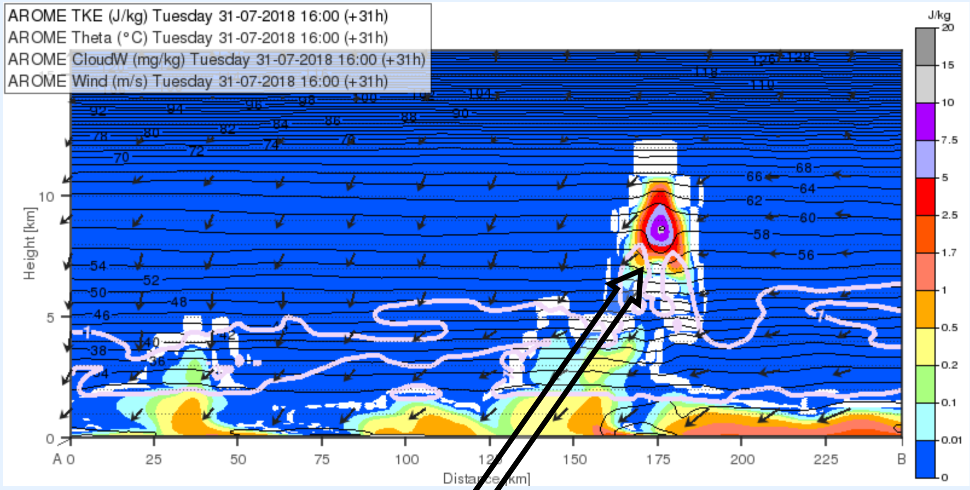
$$\varepsilon = \frac{C_\varepsilon e^{3/2}}{l_\varepsilon}$$

Eddies of smaller size are more effective in dissipating TKE to heat

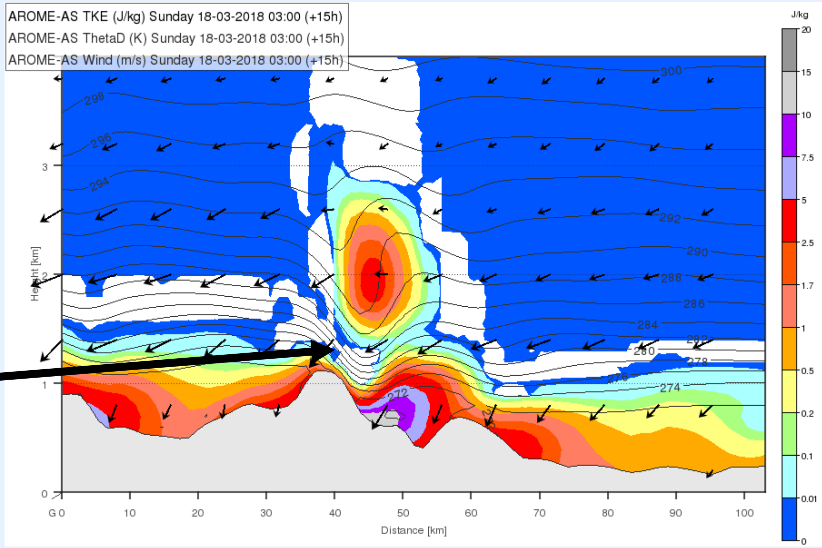
l_ε depends both on TKE and buoyancy

Other situations with usually high TKE

- Thunderstorms, tops of convective clouds
- Mountain waves
- Problems: only vertical diffusivity and isotropic turbulence expected in most of the NWP models, persistence of turbulence in stable regimes, very small scale effects, terrain-types not represented, etc.
- Bumpiness – sudden changes of flight altitude could be produced solely by wind shear/gravity waves in almost laminar flow?

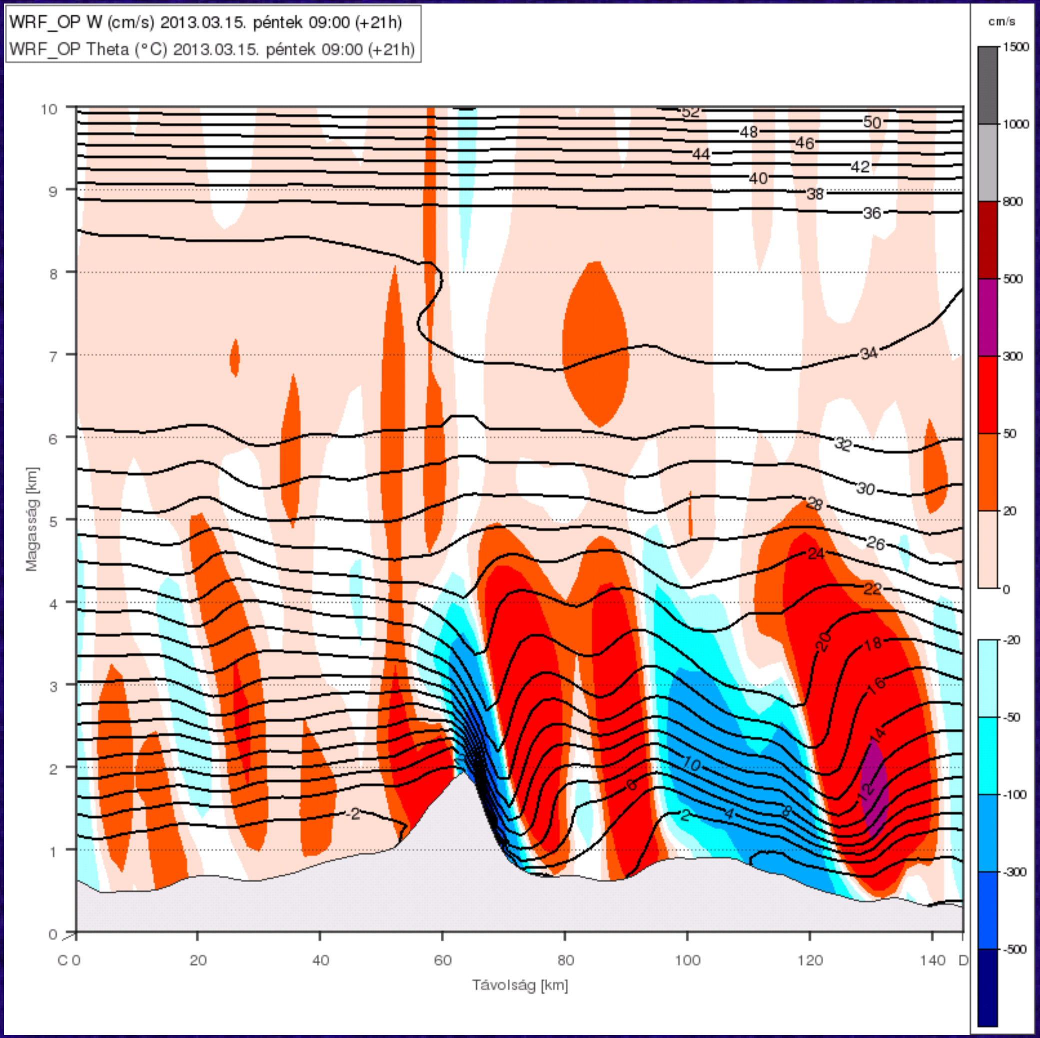


Turbulence in the rising „bubble" at the top of a simulated thunderstorm



Large-amplitude mountain wave (over central Slovakia)

Mountain waves



Simulated mountain wave with large amplitude during the High-Tatra downslope windstorm on 15 March 2013

Theoretical background

- Equation of motion (conservation of momentum)
- First law of thermodynamics (cons. of energy)
- Continuity equation (conservation of mass)

$$\rho \frac{du_i}{dt} = \rho \left(\frac{\partial u_i}{\partial t} + u_j \partial_j u_i \right) = -\rho g_i - 2\rho \varepsilon_{ijk} \Omega_j u_k - \partial_i p + \eta \partial_j \partial_j u_i + \frac{\eta}{3} \partial_i \partial_j u_j$$

$$\frac{d\theta}{dt} = \frac{\partial \theta}{\partial t} + u_j \partial_j \theta = \nu \partial_j \partial_j \theta - \frac{1}{\rho c_{pd}} \partial_j Q_j^* - \frac{L_E F}{\rho c_{pd}}$$

$$\frac{\partial \rho}{\partial t} + \partial_j \left(\rho u_j \right) = 0$$

Considerations

1. **Incompressibility** $\frac{d\rho}{dt} = 0 \Rightarrow \partial_j u_j = 0$
2. **Earth's rotation is neglected** $\Omega_j = 0$
3. **Radiation and latent heat neglected** $E = \partial_j Q_j^* = 0$
4. **Inviscideness** $\nu = 0$
5. **Stationarity** $\frac{\partial}{\partial t} = 0$
6. **Reynolds-decomposition** $A = \overline{A} + A'$
7. **Linearization (laminarity)** $A'B' = 0$
8. **Boussinesq-approximation** $\rho' \ll \rho$
9. **Hydrostatic approximation for averages** $\partial_i \overline{p} + \overline{\rho} g_i = 0$
10. **2D description (x,z plane)** $v = \frac{\partial}{\partial y} = 0$
11. **Horizontal homogeneity** $\frac{\partial \overline{A}}{\partial x} = 0$
12. **Neglecting convection in averages** $\overline{w} = 0$
13. **Exchanging density with potential temperature** $p = \rho R_d T \Rightarrow \frac{\rho'}{\overline{\rho}} = -\frac{\theta'}{\overline{\theta}}$
14. **Buoyancy, BV-frequency, reduced pressure**

$$b = g\frac{\theta'}{\overline{\theta}} \quad N^2 = \frac{g}{\overline{\theta}}\frac{\partial \overline{\theta}}{\partial z} \quad P' = \frac{p'}{\rho}$$

Non-stationary equations

- **Horizontal motion** $$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + w'\frac{\partial \bar{u}}{\partial z} + \frac{\partial P'}{\partial x} = 0$$
- **Vertical motion** $$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} + \frac{\partial P'}{\partial z} = b$$
- **Thermodynamics** $$\frac{\partial b}{\partial t} + \bar{u}\frac{\partial b}{\partial x} + w'N^2 = 0$$
- **Continuity** $$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

Stationary equations

$$\bar{u}\frac{\partial u'}{\partial x}+w'\frac{\partial \bar{u}}{\partial z}+\frac{\partial P'}{\partial x}=0$$

$$\bar{u}\frac{\partial w'}{\partial x}+\frac{\partial P'}{\partial z}=b$$

$$\bar{u}\frac{\partial b}{\partial x}+w'N^2=0$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0$$

$$(3)\rightarrow\frac{\partial b}{\partial x}=-\frac{w'N^2}{\bar{u}}$$

$$(4)\rightarrow\frac{\partial u'}{\partial x}=-\frac{\partial w'}{\partial z}$$

$$\frac{\partial(1)}{\partial z}-\frac{\partial(2)}{\partial x}\rightarrow$$

$$\left(\frac{\partial \bar{u}}{\partial z}\frac{\partial u'}{\partial x}+\bar{u}\frac{\partial^2 u'}{\partial x\partial z}+\frac{\partial w'}{\partial z}\frac{\partial \bar{u}}{\partial z}+w'\frac{\partial^2 \bar{u}}{\partial z^2}+\frac{\partial^2 P'}{\partial x\partial z}\right)$$

$$-\left(\bar{u}\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 P'}{\partial x\partial z}-\frac{\partial b}{\partial x}\right)=0$$

$$\frac{\partial \bar{u}}{\partial z}\frac{\partial u'}{\partial x}+\bar{u}\frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial x}\right)+\frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial z}+\frac{\partial^2 \bar{u}}{\partial z^2}w'$$

$$-\bar{u}\frac{\partial^2 w'}{\partial x^2}-\frac{w'N^2}{\bar{u}}=0$$

$$-\bar{u}\frac{\partial^2 w'}{\partial z^2}-\bar{u}\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial z^2}w'-\frac{N^2}{\bar{u}}w'=0$$

$$\boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{N^2}{\bar{u}^2}-\frac{1}{\bar{u}}\frac{\partial^2 \bar{u}}{\partial z^2}\right)\right]w'=0}$$

Non-stationary equations

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+w'\frac{\partial \bar{u}}{\partial z}+\frac{\partial P'}{\partial x}=0$$

$$\frac{\partial w'}{\partial t}+\bar{u}\frac{\partial w'}{\partial x}+\frac{\partial P'}{\partial z}=b$$

$$\frac{\partial b}{\partial t}+\bar{u}\frac{\partial b}{\partial x}+w'N^2=0$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0$$

$$(3)\rightarrow\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)b=-N^2w'$$

$$(4)\rightarrow\frac{\partial u'}{\partial x}=-\frac{\partial w'}{\partial z}$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial(2)}{\partial x}\rightarrow\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial w'}{\partial x}+\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2P'}{\partial x\partial z}+N^2\frac{\partial w'}{\partial x}=0$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial(1)}{\partial z}\rightarrow\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u'+\frac{\partial \bar{u}}{\partial z}w'\right]+\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2P'}{\partial z\partial x}=0$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial(1)}{\partial z}-\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial(2)}{\partial x}\rightarrow$$

$$\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial u'}{\partial z}+\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial \bar{u}}{\partial z}\frac{\partial u'}{\partial x}+\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\bar{u}}{\partial z^2}w'+\frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial z}\right)\right]-\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial w'}{\partial x}+N^2\frac{\partial w'}{\partial x}\right]=0$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial u'}{\partial z}+\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2\bar{u}}{\partial z^2}w'-\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial w'}{\partial x}=0$$

$$\frac{\partial}{\partial x}\rightarrow\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2u'}{\partial z\partial x}+\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial^2\bar{u}}{\partial z^2}\frac{\partial w'}{\partial x}-\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial^2w'}{\partial x^2}=0$$

Non-stationary equations

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2 u'}{\partial z\partial x}+\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial^2\overline{u}}{\partial z^2}\frac{\partial w'}{\partial x}-\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial^2 w'}{\partial x^2}=0$$

$$-\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2 w'}{\partial z^2}+\frac{\partial^2\overline{u}}{\partial z^2}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial w'}{\partial x}-\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial^2 w'}{\partial x^2}=0$$

$$\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)+N^2\frac{\partial^2}{\partial x^2}\right]w'=\frac{\partial^2\overline{u}}{\partial z^2}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial w'}{\partial x}$$

$$\boxed{\left[\left(\frac{1}{\overline{u}}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{N^2}{\overline{u}^2}\frac{\partial^2}{\partial x^2}-\frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}\left(\frac{1}{\overline{u}}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\right]w'=0}$$

Stationary equations from the above:

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{N^2}{\overline{u}^2}-\frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}\right)\right]w'=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{N^2}{\overline{u}^2}-\frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}\right)\right]w'=\omega_0(z)x+w_0(z)$$

Scorer-parameter $\ell^2=\frac{N^2}{\overline{u}^2}-\frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}$

Non-stationary equations

Wave equation

$$\left[\left(\frac{1}{\overline{u}}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{N^2}{\overline{u}^2}\frac{\partial^2}{\partial x^2}-\underbrace{\frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}}_{\approx 0}\left(\frac{1}{\overline{u}}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\right]w'=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\underbrace{\left(\frac{N^2}{\overline{u}^2}-\frac{1}{\overline{u}}\frac{\partial^2\overline{u}}{\partial z^2}\right)}_{\ell^2}\right]w'=0$$

Dispersion relation

$$w'(x,z)=We^{i(k_x x+k_z z-\omega t)}\Rightarrow\begin{cases}(\omega-\overline{u}k_x)^2=\dfrac{(Nk_x)^2}{k_x^2+k_z^2}\\ \ell^2=k_x^2+k_z^2\end{cases}$$

Phase and group velocity

$$c_{f_k}=\frac{\omega}{k_k};\, c_{g_k}=\frac{\partial\omega}{\partial \mathrm{k}_k}$$

$$c_{f_x}=\overline{u}\pm\frac{N}{\sqrt{k_x^2+k_z^2}}\qquad c_{g_x}=\overline{u}\pm\frac{Nk_z^2}{(k_x^2+k_z^2)^{\frac{3}{2}}}$$

$$c_{f_z}=\overline{u}\frac{k_x}{k_z}\pm\frac{Nk_x}{k_z\sqrt{k_x^2+k_z^2}}\qquad c_{g_z}=\mp\frac{Nk_xk_z}{(k_x^2+k_z^2)^{\frac{3}{2}}}$$

Wave amplitude doesn't necessarily decreases with height. Thus, c_{gz} can not be negative (else the wave would give its energy from Space). The physically correct solution corresponds to the lower signs.

Phase and group velocity

$$c_{f\,x} = \overline{u} - \frac{N}{\sqrt{k_x^2 + k_z^2}} \qquad c_{g\,x} = \overline{u} - \frac{N k_z^2}{(k_x^2 + k_z^2)^{\frac{3}{2}}}$$

$$\ell = \frac{N}{\overline{u}} = \sqrt{k_x^2 + k_z^2}$$

$$c_{f\,z} = \overline{u}\frac{k_x}{k_z} - \frac{N k_x}{k_z\sqrt{k_x^2 + k_z^2}} \qquad c_{g\,z} = \frac{N k_x k_z}{(k_x^2 + k_z^2)^{\frac{3}{2}}}$$

$$c_{f\,x} c_{g\,x} + c_{f\,z} c_{g\,z} =$$

$$= \left(\overline{u} - \frac{N}{\sqrt{k_x^2 + k_z^2}}\right)\left(\overline{u} - \frac{N k_z^2}{(k_x^2 + k_z^2)^{\frac{3}{2}}}\right) + \left(\overline{u}\frac{k_x}{k_z} - \frac{N k_x}{k_z\sqrt{k_x^2 + k_z^2}}\right)\frac{N k_x k_z}{(k_x^2 + k_z^2)^{\frac{3}{2}}}$$

$$= \overline{u}^2 - \overline{u}\frac{N k_z^2}{(k_x^2 + k_z^2)^{\frac{3}{2}}} - \overline{u}\frac{N}{\sqrt{k_x^2 + k_z^2}} + \frac{N}{\sqrt{k_x^2 + k_z^2}}\frac{N k_z^2}{(k_x^2 + k_z^2)^{\frac{3}{2}}} + \overline{u}\frac{k_x}{k_z}\frac{N k_x k_z}{(k_x^2 + k_z^2)^{\frac{3}{2}}}$$

$$- \frac{N k_x}{k_z\sqrt{k_x^2 + k_z^2}}\frac{N k_x k_z}{(k_x^2 + k_z^2)^{\frac{3}{2}}} = \overline{u}^2 - \mathrm{N}\overline{u}\frac{k_z^2}{\ell^3} - \frac{N\overline{u}}{\ell} + N^2\frac{k_z^2}{\ell^4} + N\overline{u}\frac{k_x^2}{\ell^3} - N^2\frac{k_x^2}{\ell^4}$$

$$= \overline{u}^2 - \frac{\overline{u}k_z^2}{\ell^2} - \overline{u}^2 + \frac{\overline{u}^2 k_z^2}{\ell^2} + \frac{\overline{u}^2 k_x^2}{\ell^2} - \frac{\overline{u}^2 k_x^2}{\ell^2} = 0$$

Phase and group velocities are perpendicular

General Solution

$$w'(x,0) = \overline{u}(0)\frac{dh}{dx}$$

$$\hat{w}(k_x, z) = \mathcal{F}_x\{w'(x,z)\}$$

$$\hat{h}(k_x) = \mathcal{F}_x\{h(x)\}$$

$$\frac{\partial^2 \hat{w}}{\partial z^2} + (\ell^2 - k_x^2)\hat{w} = 0$$

$$\hat{w}(k_x, 0) = ik_x\overline{u}(0)\hat{h}(k_x)$$

$$w'(x,z) = \overline{u}(z)\frac{dh(x)}{dx} * \mathcal{F}_{k_x}^{-1}\left\{e^{i\sqrt{\ell^2 - k_x^2 z}}\right\}$$

Special cases

- One-layer atmosphere, periodic terrain
  - Constant Scorer-parameter
  - Vertically evanescent and periodic monochromatic wave
  - Wave tilting
- One-layer atmosphere, isolated mountain
  - Fourier-transform of terrain height function exists
  - Superposition principle holds
  - Untrapped wave solution (localized over the mountain)
- Two-layer-atmosphere, periodic terrain
  - Two different Scorer-parameters, but constants in each layer
  - Reflection from layer boundary
  - Resonance (monochromatic trapping) and its necessary condition
- Two-layer atmosphere, isolated mountain
  - Continuous and discrete (quantized) trapping
  - Waves in the upper layer („pseudo-terrain”)
- Violation of conditions, secondary phenomena
  - Rotors (1st and 2nd type), wave breaking, downslope windstorm
  - Small-scale turbulence, moist waves, energetics

One layer, periodic terrain

Boundary conditions

$$h(x) = H \sin Kx$$

$$w'(x,0) = \bar{u}\frac{dh(x)}{dx} = \bar{u}HK \cos Kx$$

$$w'(x,\infty) = 0\ ;\ c_{g_z}(x,\infty) = \left.\frac{\partial \omega}{\partial k_z}\right|_{x,z=\infty} \geq 0$$

Solution

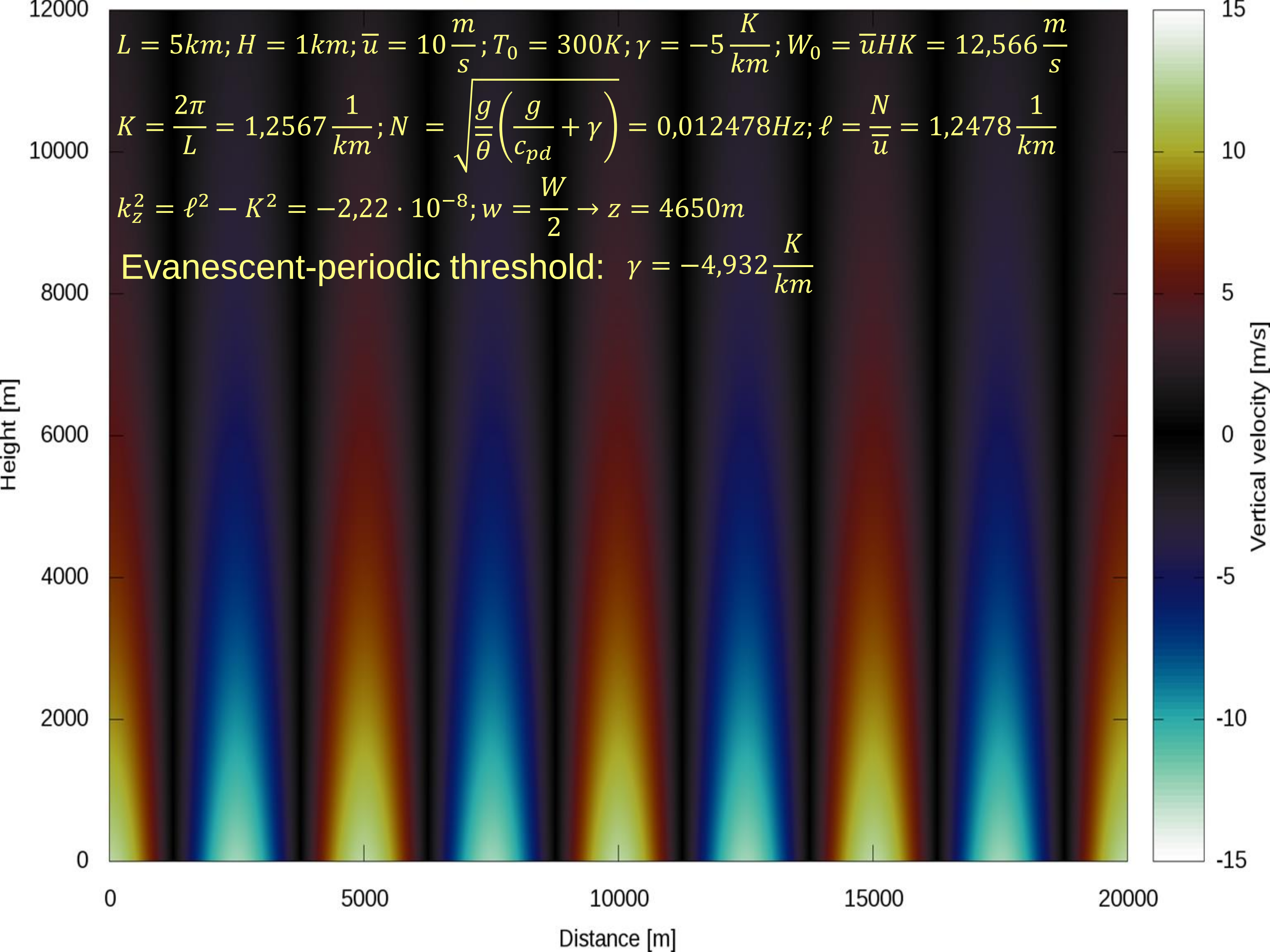
$$w'(x,z) = \Re\left(W e^{i(k_x x + k_z z)}\right) ; \ell^2 = k_x^2 + k_z^2$$

$$w'(x,0) = W\Re\left(e^{ik_x x}\right) = W \cos k_x x \overset{!}{=} \bar{u}HK \cos Kx$$

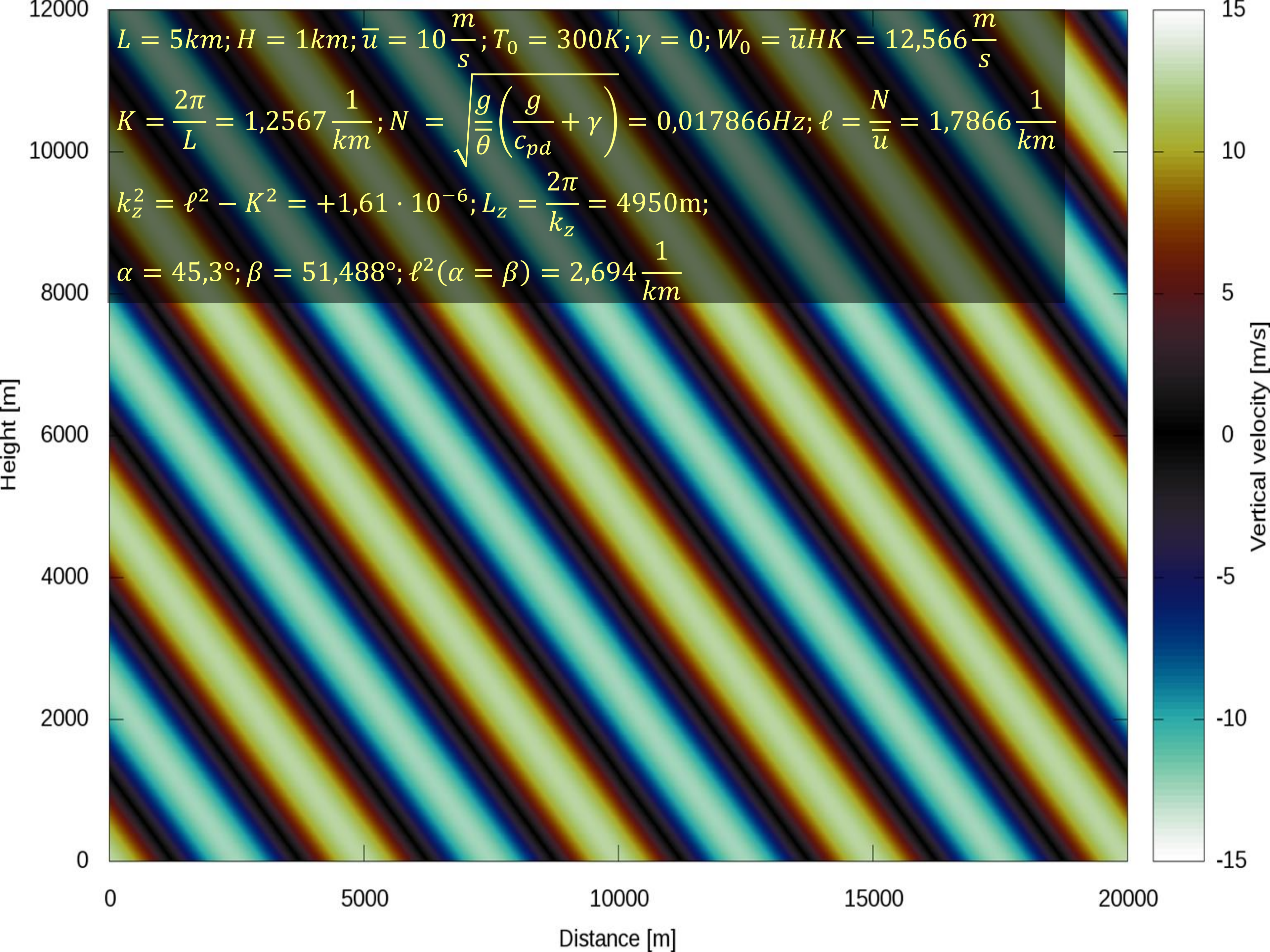
$$W = \bar{u}HK\ ;\ k_x = K \rightarrow k_z^2 = \ell^2 - K^2$$

$$w'(x,z) = \bar{u}HK \cos Kx\, e^{i\sqrt{\ell^2 - K^2} z}$$

$$\boxed{w'(x,z) = \bar{u}HK \cos Kx \begin{cases} \cos\sqrt{\ell^2 - K^2} z & \ell^2 - K^2 > 0 \\ e^{-\sqrt{K^2 - \ell^2} z} & \ell^2 - K^2 < 0 \end{cases}}$$



$L = 5km; H = 1km; \overline{u} = 10\frac{m}{s}; T_0 = 300K; \gamma = -5\frac{K}{km}; W_0 = \overline{u}HK = 12{,}566\frac{m}{s}$
$K = \frac{2\pi}{L} = 1{,}2567\frac{1}{km}; N = \sqrt{\frac{g}{\overline{\theta}}\left(\frac{g}{c_{pd}} + \gamma\right)} = 0{,}012478Hz; \ell = \frac{N}{\overline{u}} = 1{,}2478\frac{1}{km}$
$k_z^2 = \ell^2 - K^2 = -2{,}22 \cdot 10^{-8}; w = \frac{W}{2} \rightarrow z = 4650m$
Evanescent-periodic threshold: $\gamma = -4{,}932\frac{K}{km}$
Height [m]
Distance [m]
Vertical velocity [m/s]



$L = 5km; H = 1km; \overline{u} = 10\frac{m}{s}; T_0 = 300K; \gamma = 0; W_0 = \overline{u}HK = 12{,}566\frac{m}{s}$
$K = \frac{2\pi}{L} = 1{,}2567\frac{1}{km}; N = \sqrt{\frac{g}{\overline{\theta}}\left(\frac{g}{c_{pd}} + \gamma\right)} = 0{,}017866Hz; \ell = \frac{N}{\overline{u}} = 1{,}7866\frac{1}{km}$
$k_z^2 = \ell^2 - K^2 = +1{,}61 \cdot 10^{-6}; L_z = \frac{2\pi}{k_z} = 4950\text{m};$
$\alpha = 45{,}3°; \beta = 51{,}488°; \ell^2(\alpha = \beta) = 2{,}694\frac{1}{km}$
Height [m]
0
2000
4000
6000
8000
10000
12000
Distance [m]
0
5000
10000
15000
20000
Vertical velocity [m/s]
15
10
5
0
-5
-10
-15

One layer, periodic terrain

Wave tilting

$$\left(\frac{2\pi}{L_z}\right)^2 = \ell^2 - \left(\frac{2\pi}{L}\right)^2$$

$$\left(\frac{L}{L_z}\right)^2 = \left(\frac{L\ell}{2\pi}\right)^2 - 1$$

$$\operatorname{tg}\alpha = \frac{L}{L_z} = \sqrt{\left(\frac{L\ell}{2\pi}\right)^2 - 1}$$

Terrain slope

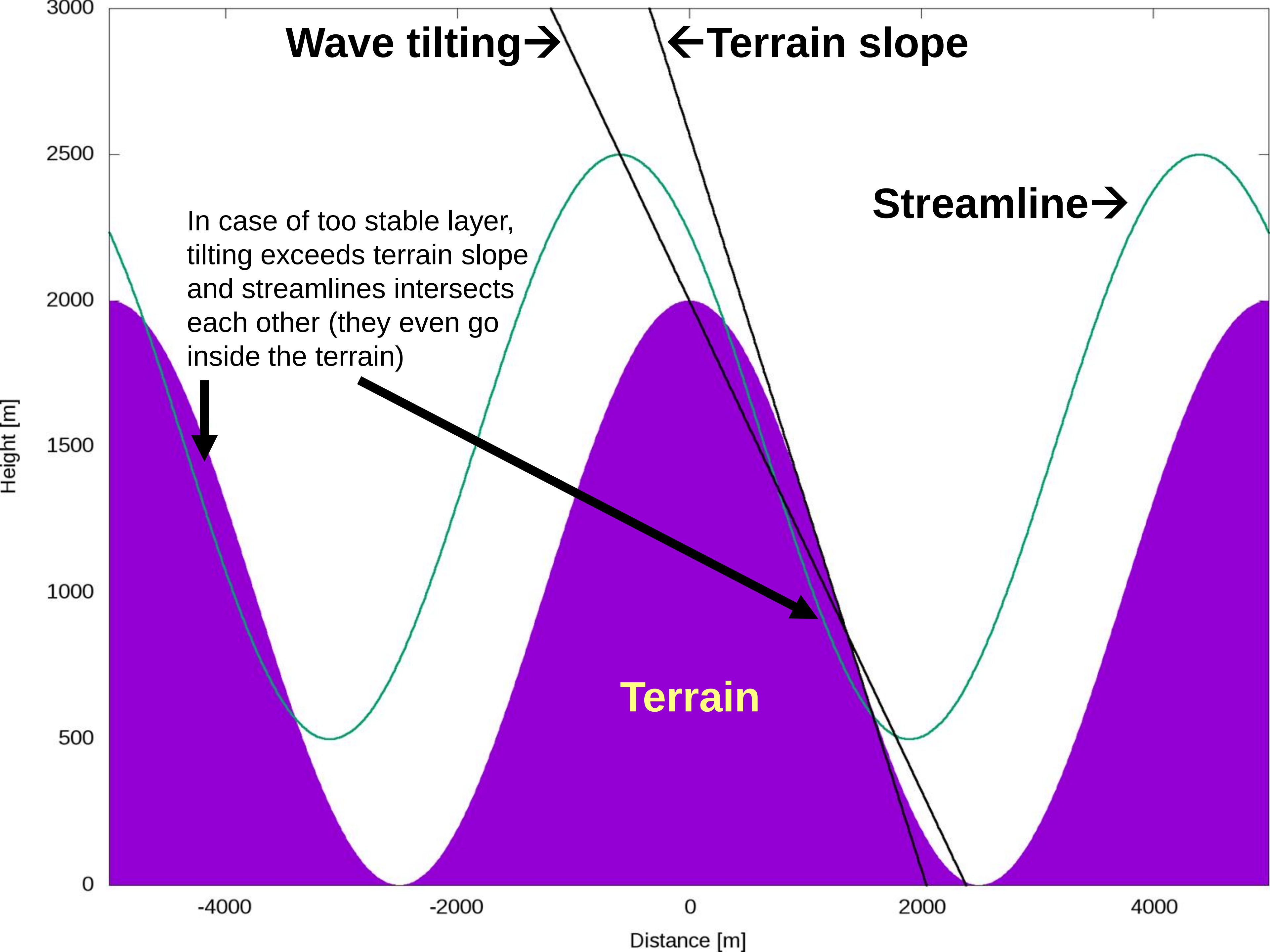
$$\operatorname{tg}\beta = \frac{W}{\overline{u}} = 2\pi\frac{H}{L}$$

$$(0 <) \operatorname{tg}\alpha \overset{!}{<} \operatorname{tg}\beta$$

Condition for temp. grad.

$$0 < \sqrt{\left(\frac{L\ell}{2\pi}\right)^2 - 1} < \frac{2\pi H}{L}$$

$$\boxed{\left(\frac{2\pi}{L}\right)^2 < \ell^2 < \left(\frac{2\pi}{L}\right)^2 \left[\left(2\pi\frac{H}{L}\right)^2 + 1\right]}$$



Wave tilting→
←Terrain slope
Streamline→
In case of too stable layer, tilting exceeds terrain slope and streamlines intersects each other (they even go inside the terrain)
Terrain
Height [m]
Distance [m]
3000
2500
2000
1500
1000
500
0
-4000
-2000
0
2000
4000

One layer, isolated mountain

$$w'(x,0) = \overline{u}\frac{dh}{dx}$$

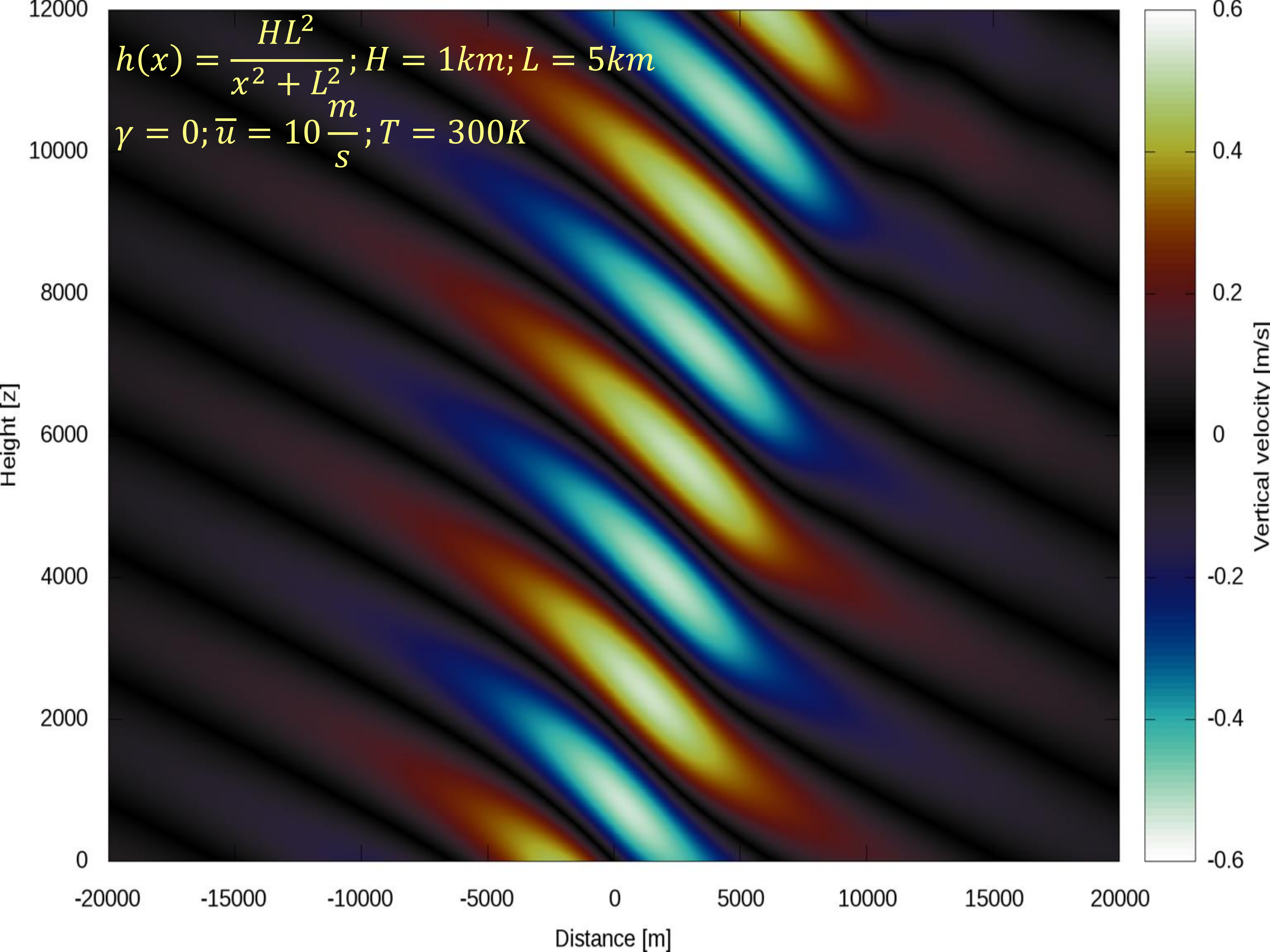
$$\widehat{w}(k_x, z) = \mathcal{F}_x\{w'(x,z)\}$$

$$\hat{h}(k_x) = \mathcal{F}_x\{h(x)\}$$

$$\frac{\partial^2 \widehat{w}}{\partial z^2} + (\ell^2 - k_x^2)\widehat{w} = 0$$

$$\widehat{w}(k_x, 0) = ik_x\overline{u}\hat{h}(k_x)$$

$$w'(x,z) = \overline{u}\frac{dh(x)}{dx} * \mathcal{F}_{k_x}^{-1}\left\{e^{i\sqrt{\ell^2 - k_x^2}z}\right\}$$



$h(x) = \frac{HL^2}{x^2 + L^2}; H = 1km; L = 5km$
$\gamma = 0; \overline{u} = 10\frac{m}{s}; T = 300K$
Height [z]
Distance [m]
Vertical velocity [m/s]
0
2000
4000
6000
8000
10000
12000
-20000
-15000
-10000
-5000
0
5000
10000
15000
20000
0.6
0.4
0.2
0
-0.2
-0.4
-0.6

Two layer, periodic terrain

- **Constant, but different Scorer-parameters in the two layer**
  - Note the variables with an *L* index in the lower layer, and an *U* index in the upper layer
- **Phenomena similar to those known in optics**
  - Reflection at the upper boundary in the lower layer
  - Transmission into the upper layer (but its treated as a one-layer with a terrain made of the layer boundary)
- **Boundary condition at layer boundary (where *z*=*Z*)**
  - Reflection coefficient (*r*)

$$\frac{w_L'(x,Z)}{\overline{u}_L(Z)} = \frac{w_U'(x,Z)}{\overline{u}_U(Z)}$$

Two layer, periodic terrain

- Wave reflection
  - Vertical group velocity changes sign
  - At layer boundary *r*<1, at ground *r*=1
  - The sum of them results in 'chessboard pattern'
  - Infinitely many reflections happen

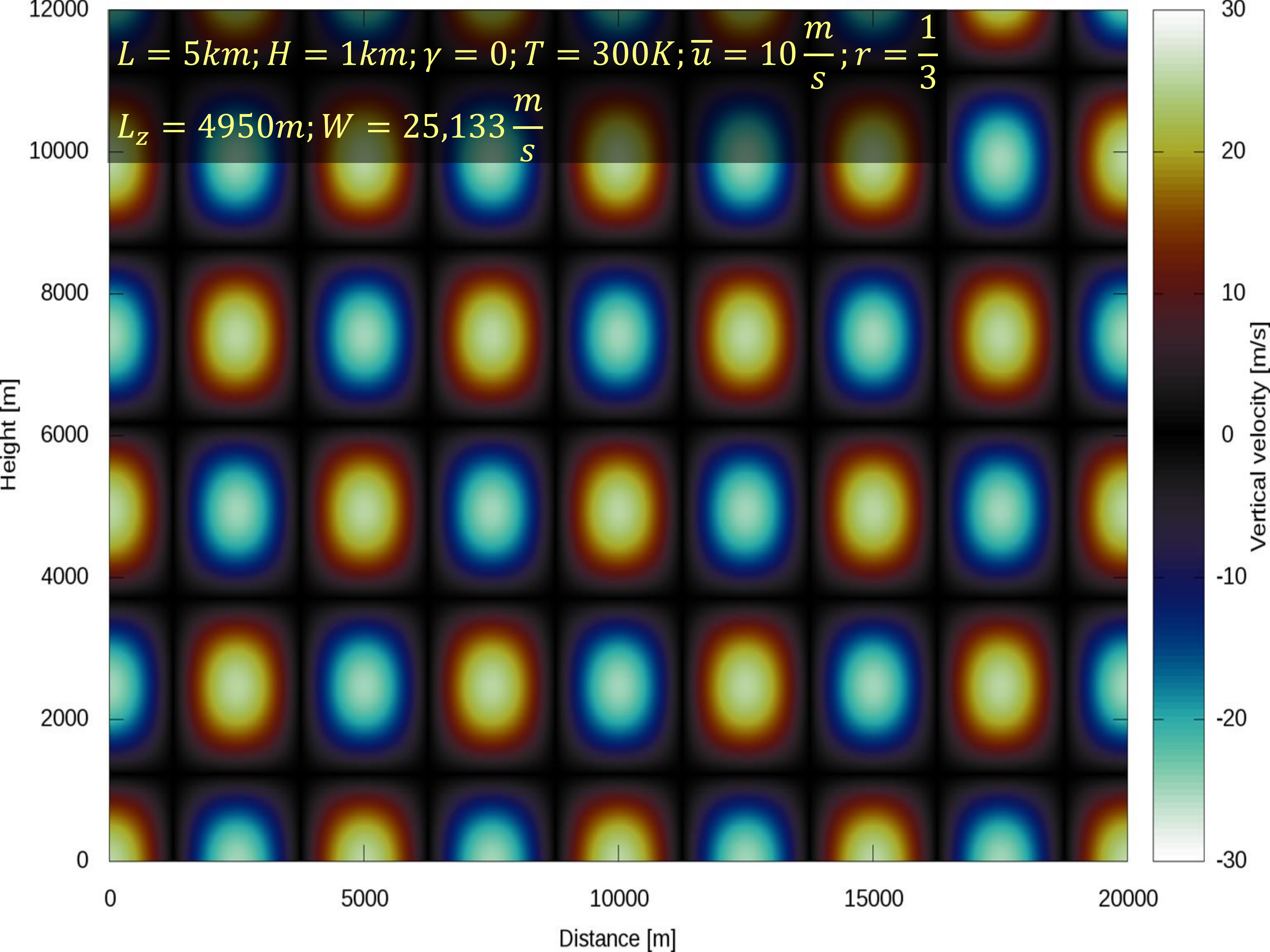
$$w_n'(x,z) = 2W_n e^{iKx}\cos\sqrt{\ell^2 - K^2}z$$

$$w'(x,z) = \sum_{n=1}^{\infty} w_n(x,z) = \sum_{n=1}^{\infty} 2W_n e^{iKx}\cos\sqrt{\ell^2 - K^2}z$$

$$W_n = rW_{n-1}; W_1 = 2rW_0$$

$$W = \sum_{n=1}^{\infty} W_n = \frac{W_1}{1-r} = \frac{2r}{1-r}W_0$$

To hold lower boundary condition: $\frac{2r}{1-r} = 1 \rightarrow r = \frac{1}{3}$



$L = 5km; H = 1km; \gamma = 0; T = 300K; \overline{u} = 10\frac{m}{s}; r = \frac{1}{3}$
$L_z = 4950m; W = 25,133\frac{m}{s}$
Height [m]
Distance [m]
Vertical velocity [m/s]
0
2000
4000
6000
8000
10000
12000
5000
15000
20000
30
20
10
-10
-20
-30

Two layer, periodic terrain

- The reflected wave
  - Waves evanescent in the upper layer are reflected fully from the layer boundary in the lower layer
  - Full reflection (r=1) results formally in infinite amplitude
    - Violation of considerations (eg. that perturbation is small etc) and lower boundary condition
    - Secondary phenomena appear which keep the amplitude finite by dissipating some energy from the wave
  - Advantages of the infinite amplitude:
    - Every mode with r<1 (including W_0) can be neglected
    - Reflection coefficient depends also on the local amplitude (or i.e. the phase) at the layer boundary (the local relative to the full amplitude appears as a multiplicator)
  - Full reflection is possible only when a constraint of Scorer-parameters holds

Two layer, periodic terrain

- **Conditions of full reflection (resonance)**
  - **Let *r* be 1**

$$W_{n_L} \cos k_{x_L} x\, e^{ik_{z_L}Z} = W_{n_U} \cos k_{x_U} x\, e^{ik_{z_U}Z}$$

$$W_{n_U} = W_{n-1_L}(1-r) = 2\overline{u}_L HKr^{n-1}(1-r)$$

$$W_{n_L} = 2\overline{u}_L HKr^n$$

$$k_{x_U} = k_{x_L} = K; k_x^2 + k_z^2 = \ell^2$$

$$2\overline{u}_L HKr^n \cos Kx\, e^{i\sqrt{\ell_L^2 - K^2}Z} = 2\overline{u}_U HKr^{n-1} \cos Kx\, e^{i\sqrt{\ell_U^2 - K^2}Z}$$

$$\frac{1}{r} - 1 = \frac{\overline{u}_U}{\overline{u}_L} \frac{\cos\sqrt{\ell_L^2 - K^2}Z}{\cos\sqrt{\ell_U^2 - K^2}Z}$$

$$r = 1 \Rightarrow \cos\sqrt{\ell_L^2 - K^2} = 0 \rightarrow K^2 = \ell_L^2 - \left[\left(j + \frac{1}{2}\right)\frac{\pi}{Z}\right]^2$$

Two layer, periodic terrain

- **Conditions of full reflection (resonance)**
  - Let the wave be evanescent in the upper layer

$$\ell_U^2 - K^2 < 0$$

$$\ell_U^2 - \left(\ell_L^2 - \left[\left(j + \frac{1}{2} \right) \frac{\pi}{Z} \right]^2 \right) < 0$$

$$\Downarrow$$

$$\ell_L^2 - \ell_U^2 > \left(j + \frac{1}{2} \right)^2 \frac{\pi^2}{Z^2}$$

$$\boxed{\ell_L^2 - \ell_U^2 > \left(\frac{\pi}{2Z} \right)^2}$$

Two layer, isolated mountain

- **Only a multiplier containing the reflection modifies the one-layer solution**
  - **Reflection can depend on the wavenumber**

$$\hat{w}(k_x, z) = \left(\frac{2r(k_x)}{1 - r(k_x)} + 1\right) ik_x \overline{u}(z)\hat{h}(k_x)$$

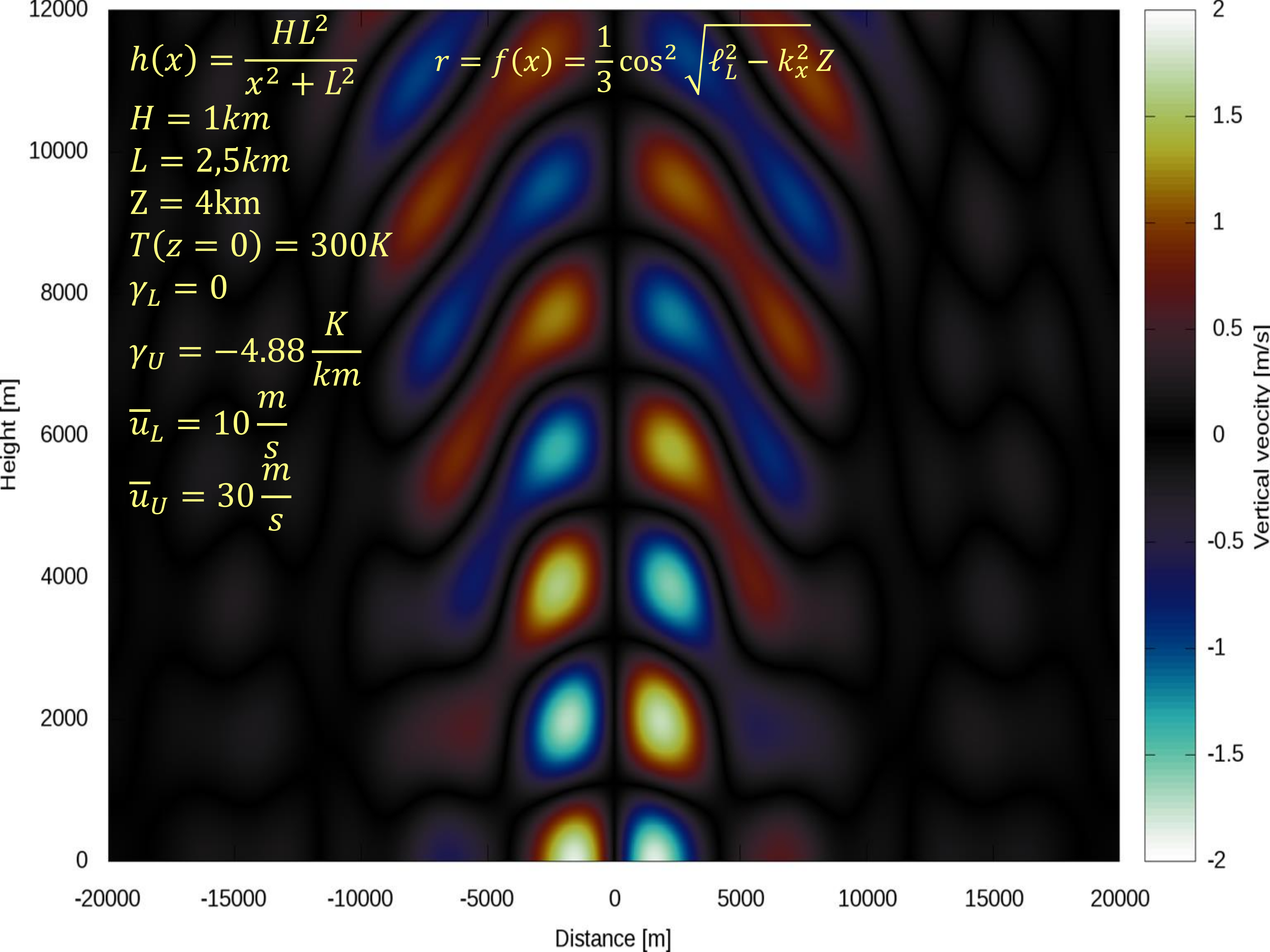
$$w'(x, z) = \overline{u}(z)\frac{dh(x)}{dx} * \mathcal{F}_{k_x}^{-1}\left\{\left(\frac{2r(k_x)}{1 - r(k_x)} + 1\right)\cos\sqrt{\ell^2 - k_x^2 z}\right\}$$

Two layer, isolated mountain

- **Applying conditions of full reflection**
  - **Fourier-integral only between Scorer-params.**

$$w'(x,z) = \frac{\overline{u}(z)}{\sqrt{2\pi}} \frac{dh(x)}{dx} * \int_{k_x=\ell_U}^{\ell_L} \left(\frac{2r(k_x)}{1-r(k_x)} + 1 \right) \cos \sqrt{\ell_L^2 - k_x^2} z \cdot e^{ik_x x} dk_x$$

$$= \frac{\overline{u}(z)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dh(x')}{dx'} \int_{\ell_U}^{\ell_L} \left(\frac{2r(k_x)}{1-r(k_x)} + 1 \right) \cos \sqrt{\ell_L^2 - k_x^2} z \cdot e^{ik_x (x-x')} dk_x \, dx'$$



$h(x) = \frac{HL^2}{x^2 + L^2}$
$r = f(x) = \frac{1}{3}\cos^2\sqrt{\ell_L^2 - k_x^2}\,Z$
$H = 1km$
$L = 2{,}5km$
$\mathrm{Z} = 4\mathrm{km}$
$T(z = 0) = 300K$
$\gamma_L = 0$
$\gamma_U = -4.88\frac{K}{km}$
$\overline{u}_L = 10\frac{m}{s}$
$\overline{u}_U = 30\frac{m}{s}$
Height [m]
Distance [m]
Vertical veocity [m/s]

Two layer, isolated mountain

- **Resonance, quantization**
  - **Reflection is full only when the local amplitude on the layer boundary is maximal (i.e. their phase there is *jπ*).**

$$j\frac{L_{z_L}}{2} = Z \rightarrow k_{z_L}Z = j\pi$$

$$0 < j < \frac{Z}{\pi}\sqrt{\ell_L^2 - \ell_U^2}$$

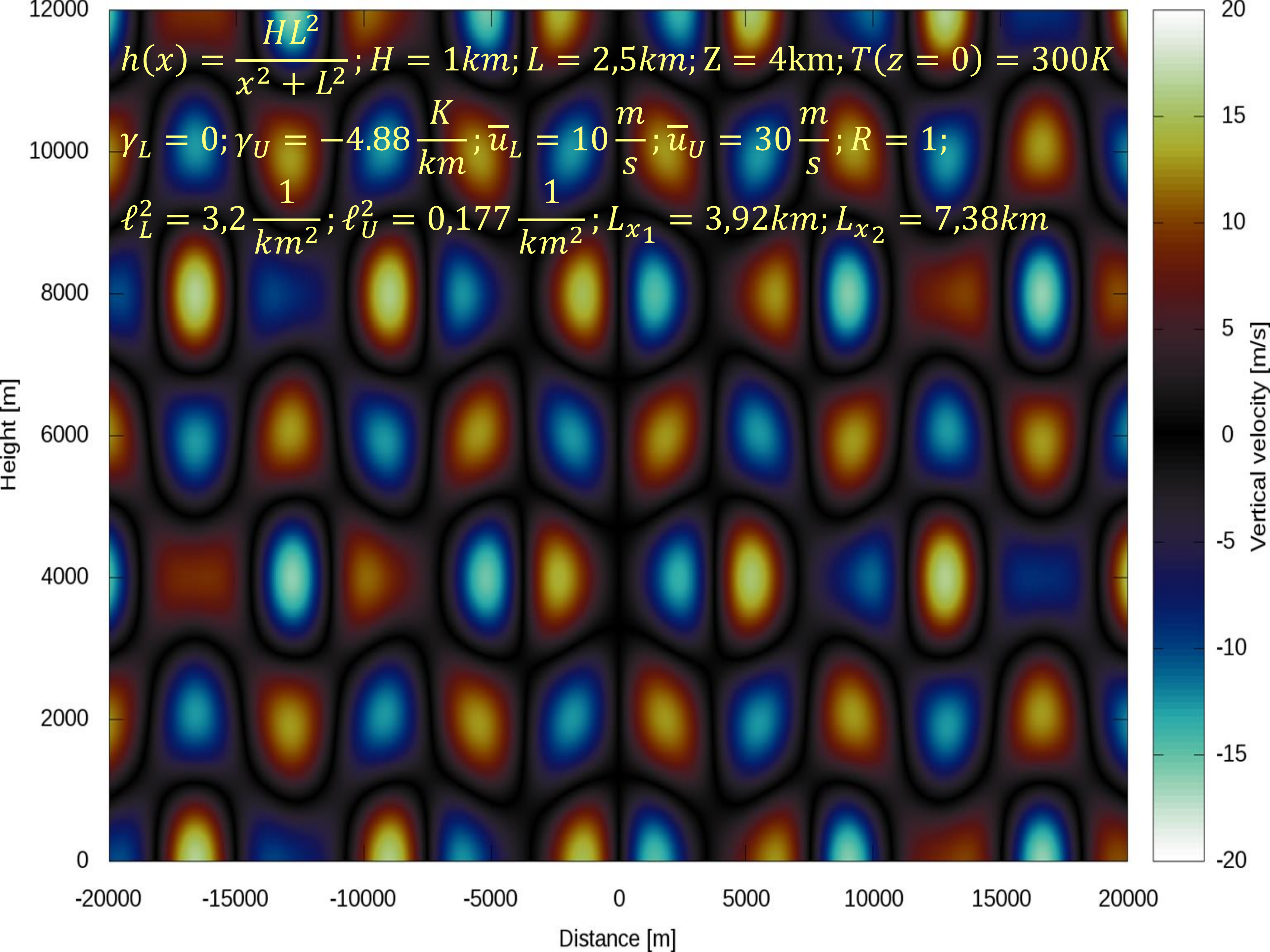
$$\ell_U^2 < {k_{x}}_j^2 = \ell_L^2 - \left(\frac{j\pi}{Z}\right)^2 < \ell_L^2$$

$$j > 0 \Rightarrow \ell_L^2 - \ell_U^2 > \left(\frac{\pi}{Z}\right)^2$$

$$R \coloneqq \frac{2r_j}{1 - r_j} + 1 < \infty \forall j$$

$$w'(x,z) = \frac{\overline{u}(z)R}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dh(x')}{dx'} \sum_{\ell_U < {k_x}_j}^{{k_x}_j < \ell_L} \cos\frac{j\pi z}{Z} e^{i\sqrt{\ell_L^2 - \left(\frac{j\pi}{Z}\right)^2}(x - x')} dx'$$

$$\boxed{w'(x,z) = \frac{\overline{u}(z)R}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dh(x')}{dx'} \sum_{j=1}^{\left\lfloor \frac{Z}{\pi}\sqrt{\ell_L^2 - \ell_U^2} \right\rfloor} \cos\frac{j\pi z}{Z} e^{i\sqrt{\ell_L^2 - \left(\frac{j\pi}{Z}\right)^2}(x - x')} dx'}$$


$h(x) = \frac{HL^2}{x^2 + L^2}; H = 1km; L = 2{,}5km; \mathrm{Z} = 4\mathrm{km}; T(z=0) = 300K$
$\gamma_L = 0; \gamma_U = -4.88\frac{K}{km}; \bar{u}_L = 10\frac{m}{s}; \bar{u}_U = 30\frac{m}{s}; R = 1;$
$\ell_L^2 = 3{,}2\frac{1}{km^2}; \ell_U^2 = 0{,}177\frac{1}{km^2}; L_{x_1} = 3{,}92km; L_{x_2} = 7{,}38km$
Height [m]
12000
10000
8000
6000
4000
2000
0
-20000
-15000
-10000
-5000
0
5000
10000
15000
20000
Distance [m]
20
15
10
5
0
-5
-10
-15
-20
Vertical velocity [m/s]

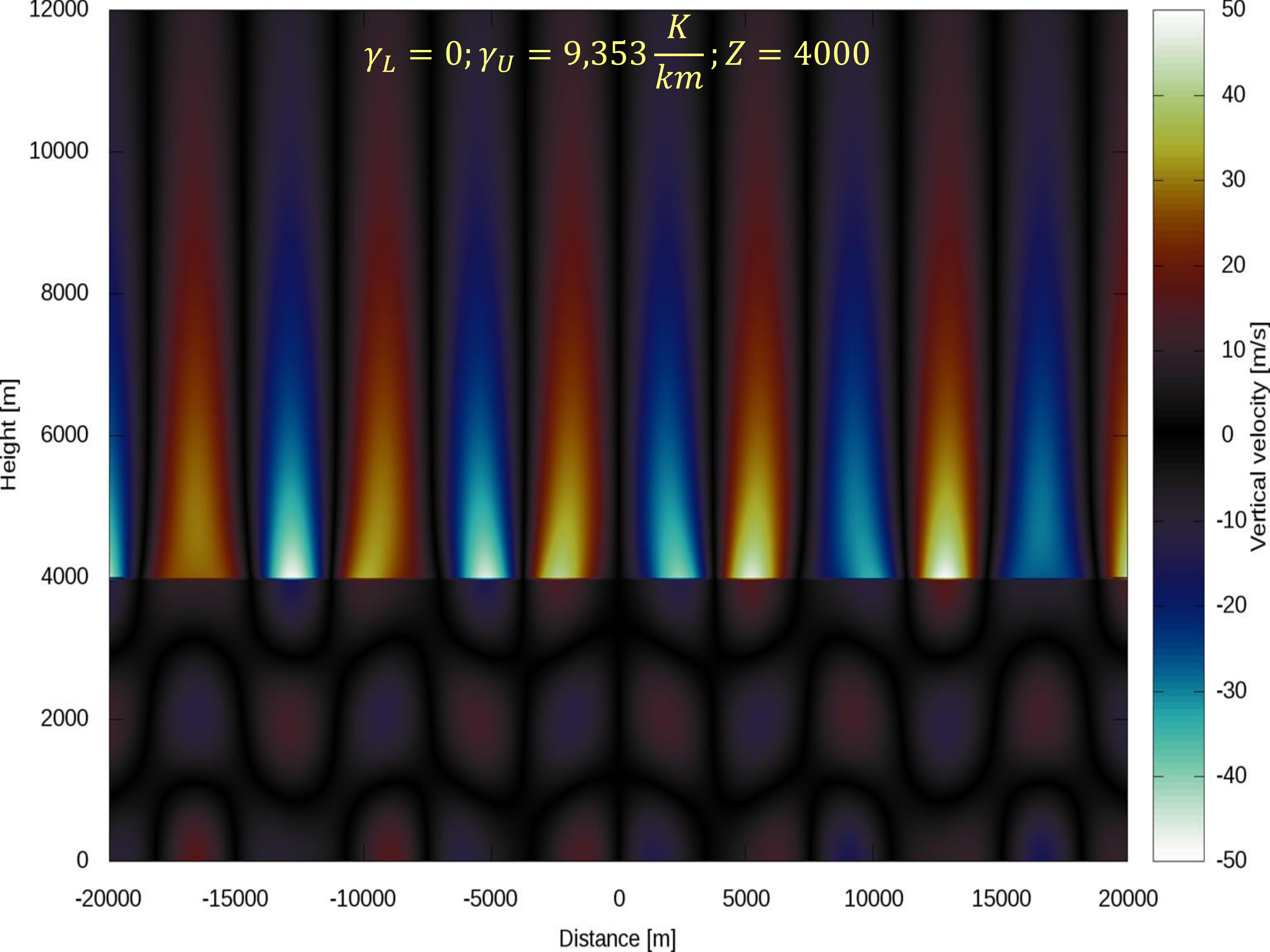
Two layer, isolated mountain

- **The layer boundary, affected by the lower layer waves, behaves as a ‘terrain’ for the upper layer**

$$w'_U(x,Z) = \frac{\overline{u}_U}{\overline{u}_L} w'_L(x,Z) \rightarrow \widehat{w}_U(k_x,Z) = \frac{\overline{u}_U}{\overline{u}_L} \widehat{w}_L(k_x,Z)$$

$$w'_U(x,z) = \overline{u}_U(z) R \frac{dh(x)}{dx} * \mathcal{F}^{-1}_{k_x}\left\{\left(\cos\sqrt{\ell_L^2 - k_x^2} Z \cdot e^{i\sqrt{\ell_U^2 - k_x^2}(z-Z)}\right)\right\}$$

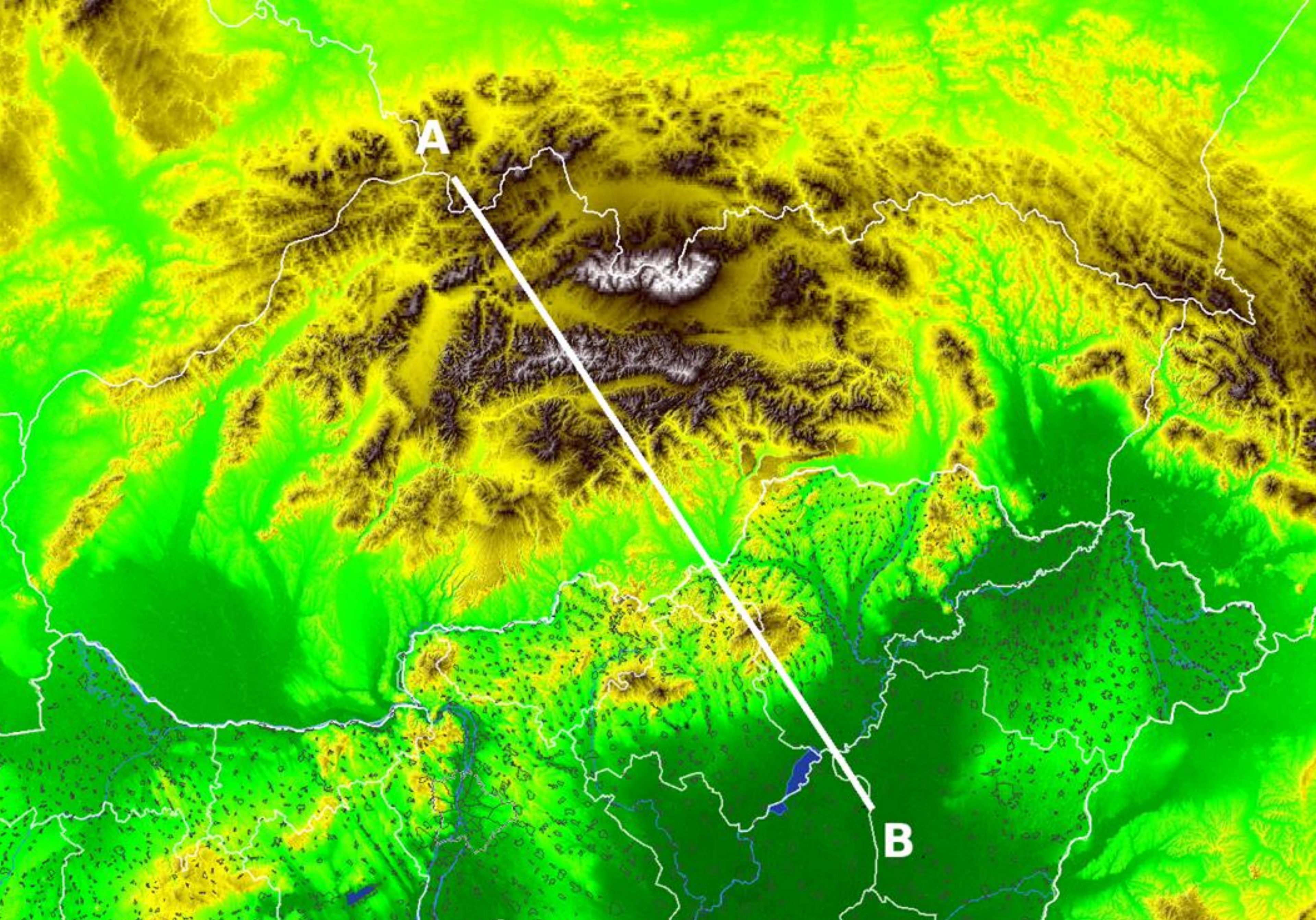
$$\boxed{w'_U(x,z) = \frac{\overline{u}_U(z) R}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dh(x')}{dx'} \sum_{j=1}^{\left\lfloor \frac{Z}{\pi}\sqrt{\ell_L^2 - \ell_U^2} \right\rfloor} (-1)^j e^{i\sqrt{\ell_U^2 - \ell_L^2 + \left(\frac{j\pi}{Z}\right)^2}} e^{i\sqrt{\ell_L^2 - \left(\frac{j\pi}{Z}\right)^2}(x-x')} dx'}$$



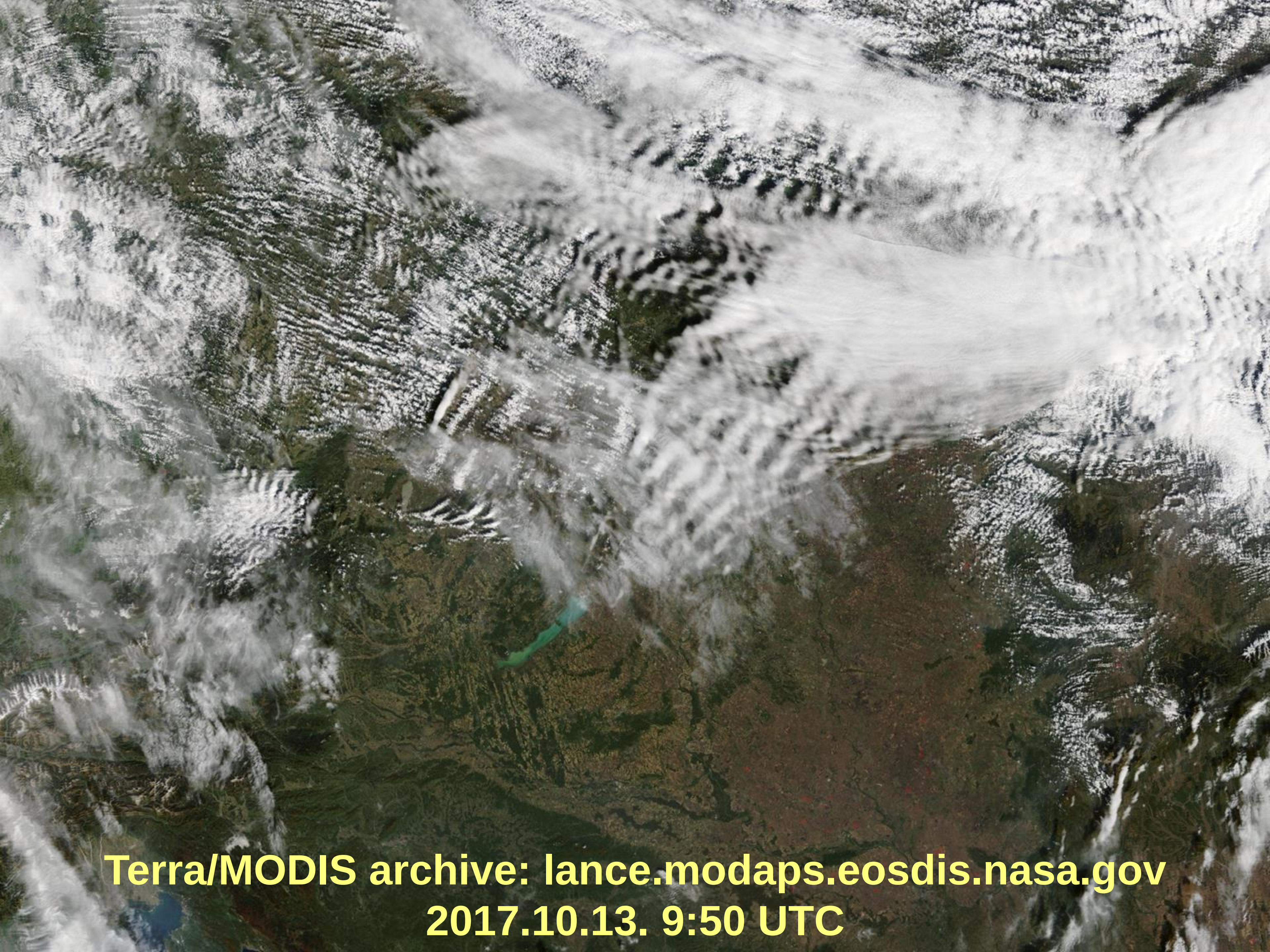
$\gamma_L = 0; \gamma_U = 9{,}353\,\frac{K}{km}; Z = 4000$
Height [m]
12000
10000
8000
6000
4000
2000
0
-20000
-15000
-10000
-5000
0
5000
10000
15000
20000
Distance [m]
Vertical velocity [m/s]
50
40
30
20
10
0
-10
-20
-30
-40
-50

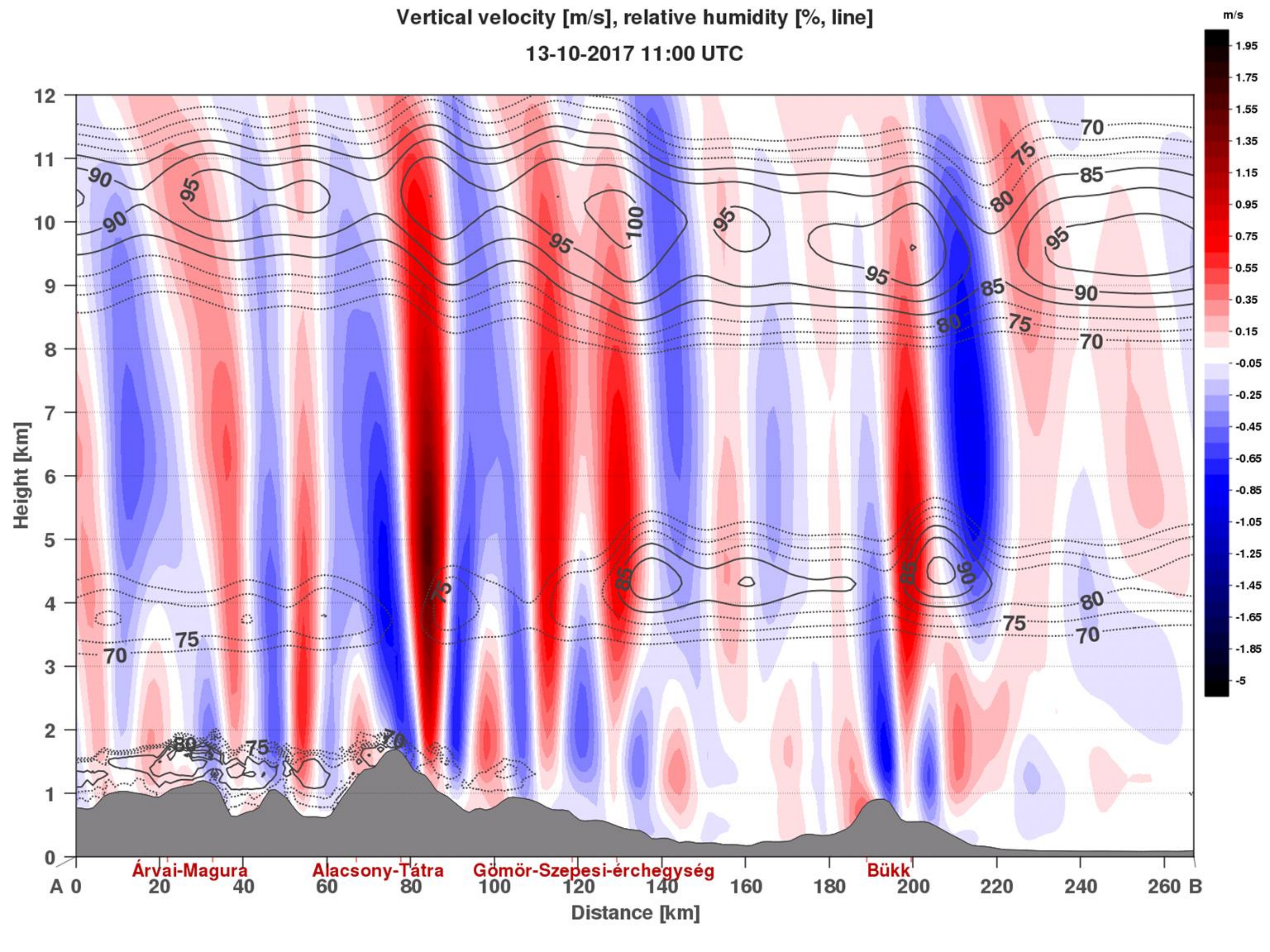
Theoretical results

- Scorer-parameter represents the wavenumber of mountain waves and characterizes their onset.
- Over a periodic terrain, the waves are tilted upstream.
- In a one layer atmosphere over an isolated mountain, the waves are localized over the mountain.
- Trapped and untrapped waves can exist in the lower layer of a two layer atmosphere.
- Trapped waves are those which are evanescent in the upper layer.
- For trapping in a two layer atmosphere, the difference between the Scorer-parameters has to exceed a threshold which decreases with the depth of the lower layer.
- Number of trapped wave modes are finite, especially usually 1-2, but can be several more if the stable (trapping) layer is several kms deep.

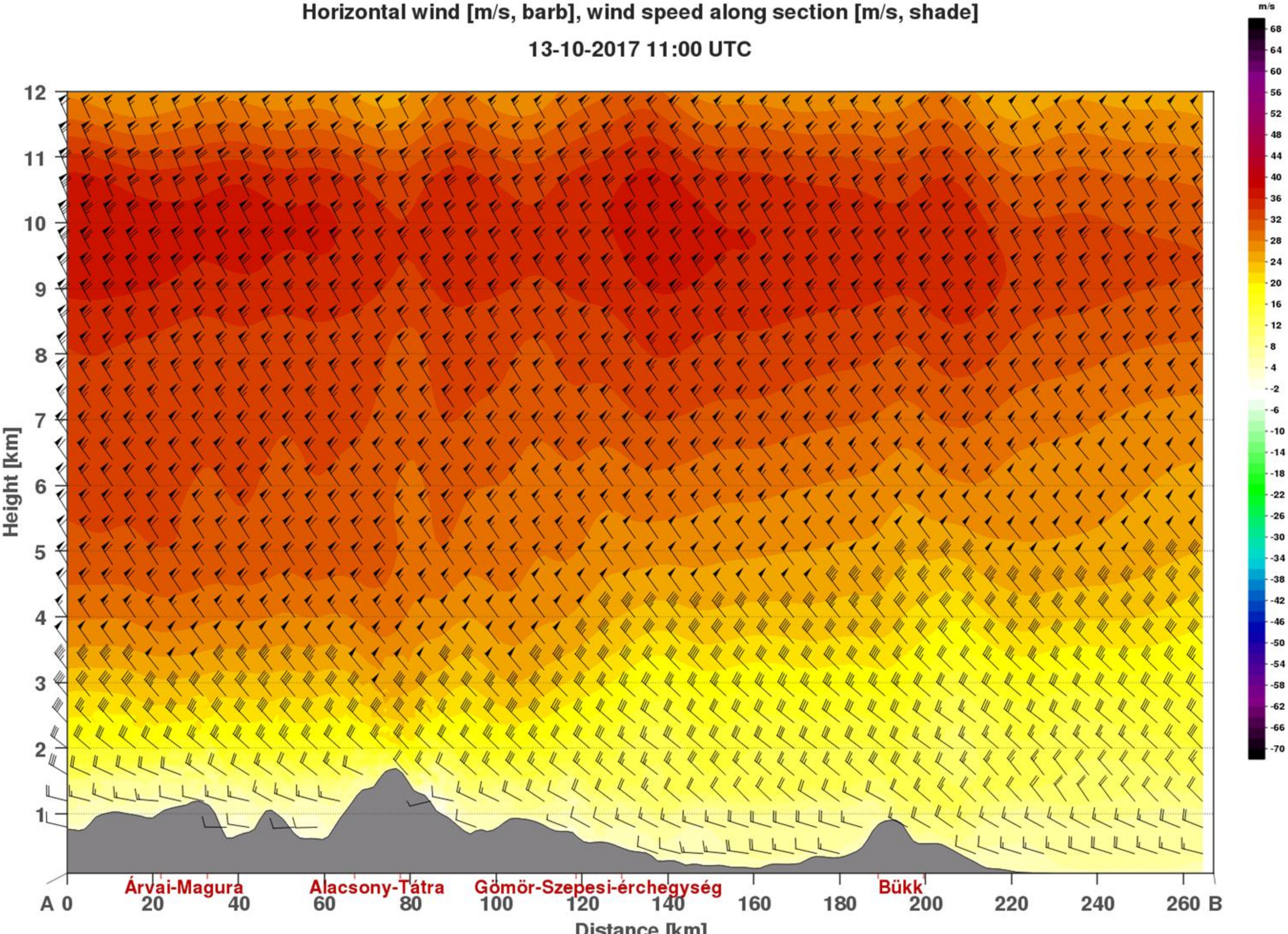


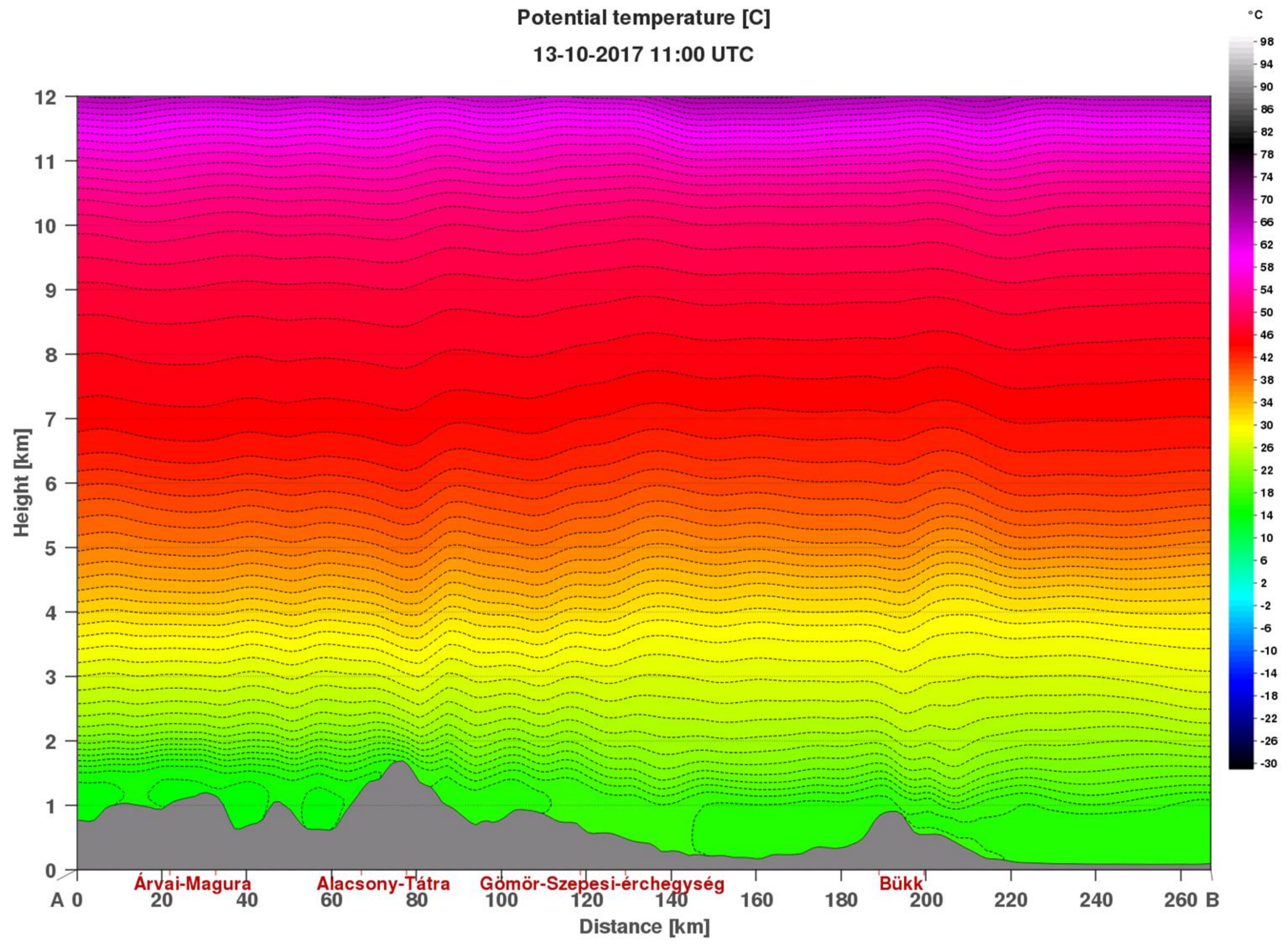
Case study: 2017.10.13. 11UTC (using AROME model)


Terra/MODIS archive: lance.modaps.eosdis.nasa.gov
2017.10.13. 9:50 UTC

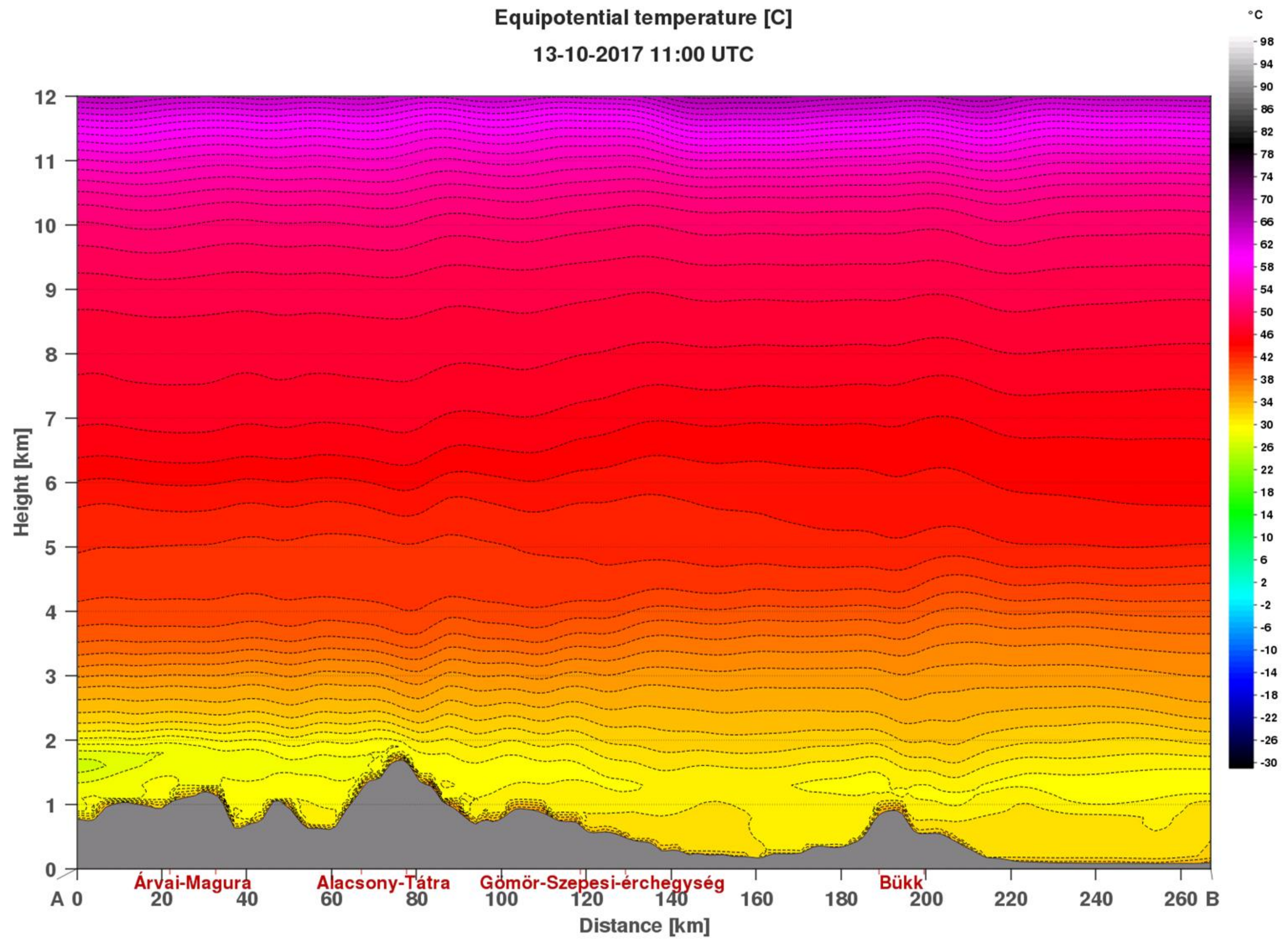


Vertical velocity [m/s], relative humidity [%, line]
13-10-2017 11:00 UTC
m/s
1.95
1.75
1.55
1.35
1.15
0.95
0.75
0.55
0.35
0.15
-0.05
-0.25
-0.45
-0.65
-0.85
-1.05
-1.25
-1.45
-1.65
-1.85
-5
Height [km]
Distance [km]
A
B
Árvai-Magura
Alacsony-Tátra
Gömör-Szepesi-érchegység
Bükk

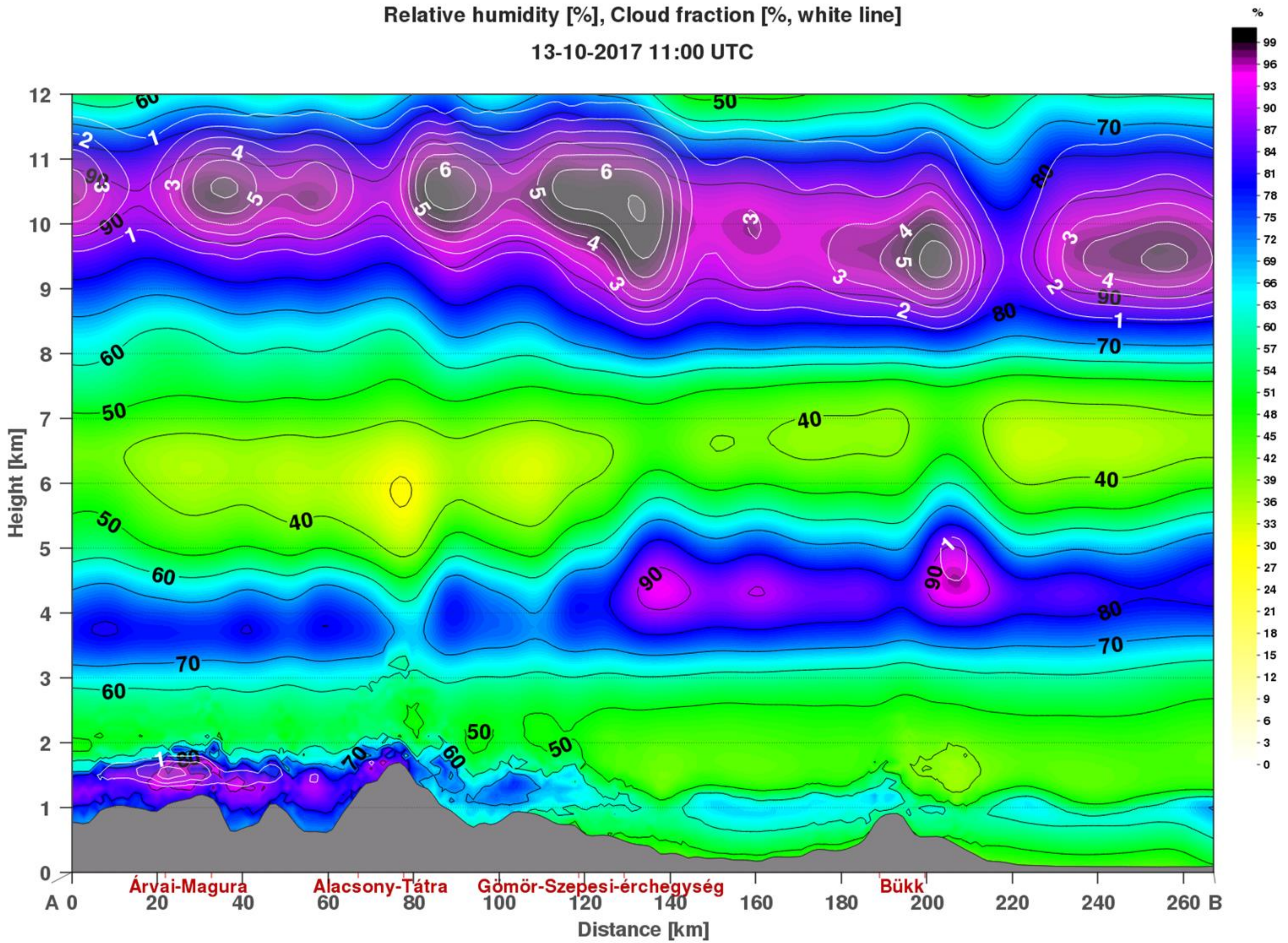

Horizontal wind [m/s, barb], wind speed along section [m/s, shade]
13-10-2017 11:00 UTC
Height [km]
Distance [km]
A
B
Árvai-Magura
Alacsony-Tátra
Gömör-Szepesi-érchegység
Bükk
m/s

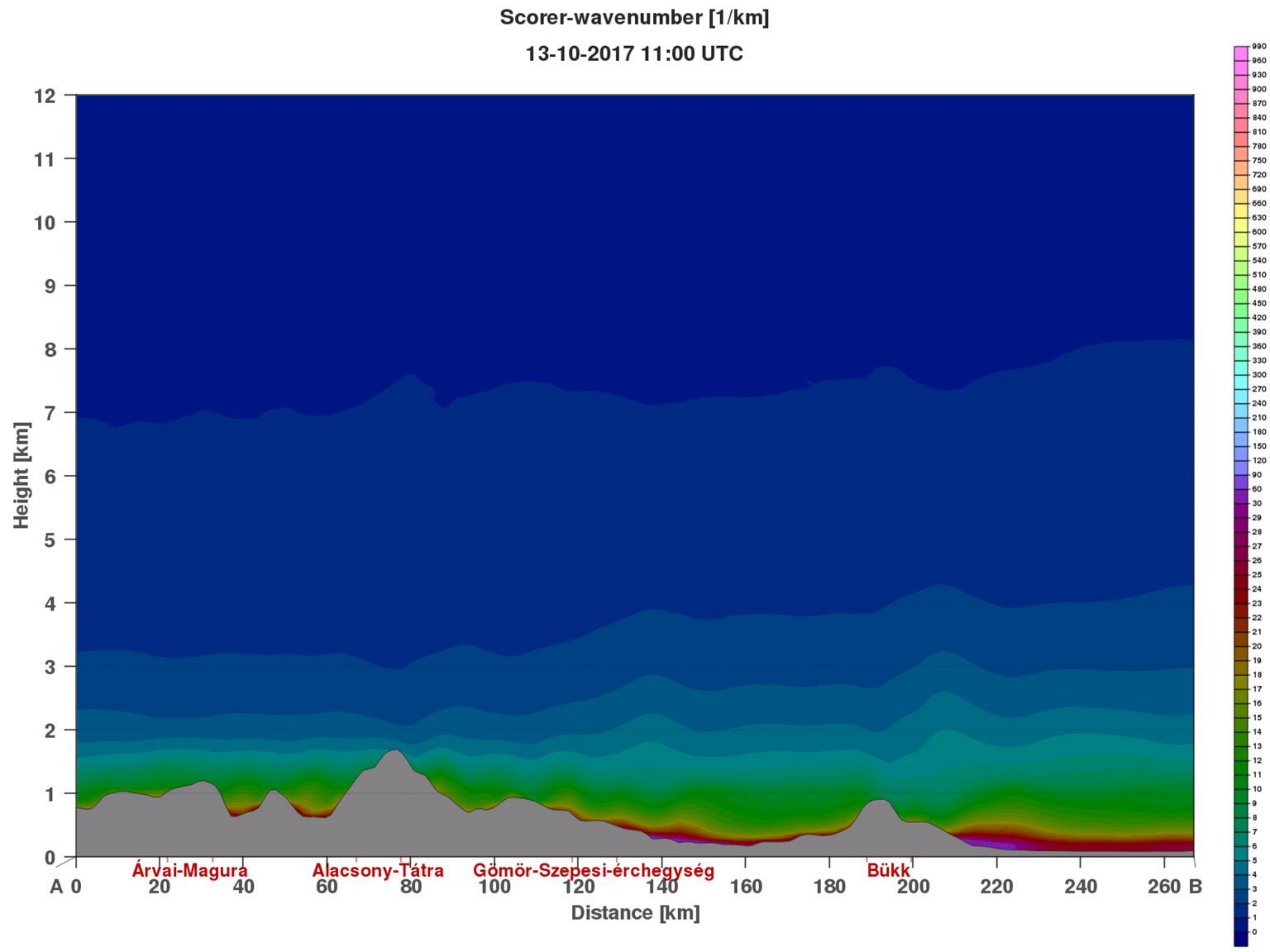


Potential temperature [C]
13-10-2017 11:00 UTC
°C
98
94
90
86
82
78
74
70
66
62
58
54
50
46
42
38
34
30
26
22
18
14
10
6
2
-2
-6
-10
-14
-18
-22
-26
-30
Height [km]
12
11
10
9
8
7
6
5
4
3
2
1
0
Árvai-Magura
Alacsony-Tátra
Gömör-Szepesi-érchegység
Bükk
A
0
20
40
60
80
100
120
140
160
180
200
220
240
260
B
Distance [km]



Equipotential temperature [C]
13-10-2017 11:00 UTC
°C
98
94
90
86
82
78
74
70
66
62
58
54
50
46
42
38
34
30
26
22
18
14
10
6
2
-2
-6
-10
-14
-18
-22
-26
-30
Height [km]
0
1
2
3
4
5
6
7
8
9
10
11
12
Árvai-Magura
Alacsony-Tátra
Gömör-Szepesi-érchegység
Bükk
A
0
20
40
60
80
100
120
140
160
180
200
220
240
260
B
Distance [km]


Relative humidity [%], Cloud fraction [%, white line]
13-10-2017 11:00 UTC
Height [km]
Distance [km]
A
B
Árvai-Magura
Alacsony-Tátra
Gömör-Szepesi-érchegység
Bükk
%



Scorer-wavenumber [1/km]
13-10-2017 11:00 UTC
Height [km]
Distance [km]
Árvai-Magura
Alacsony-Tátra
Gömör-Szepesi-érchegység
Bükk
A
B

Summary, further plans

- Mountain waves are the only phenomena which can cause vertical streaming nearly as strong as convection
- Forecasting waves (based on Scorer-param.) can be easy
- Forecasting wave turbulence is very difficult
  - Lack of complete theory (NWP modelling took place in ‘70s).
  - Major underestimation of turbulence in NWP models.
- Risk for glider pilots
  - Waves are most useful for soaring but the turbulence (with that it is often CAT and can not be forecasted) put pilots on high danger when flying in waves.
- Theoretical works are planned for describing free-atmospheric turbulence using AROME model.
  - Results of recent numerical studies on TKE modelling is under investigation in AROME to study they performance in mountain wave situations (results are still uncertain for jet-related CAT).

Thank you for your attention!

Acknowledgements

To Viktória Homonnai (OMSZ) for providing us AROME NWP and TKE data and Máté Mester (OMSZ) for sharing us his experiencies and pilot point of view